LINEAR ALGEBRA
A GEOMETRIC APPROACH

second edition

Theodore Shifrin
Malcolm R. Adams

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A Geometric Approach *second edition*

Theodore Shifrin
Malcolm R. Adams

University of Georgia

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PREFACE

One of the most enticing aspects of mathematics, we have found, is the interplay of ideas from seemingly disparate disciplines of the subject. Linear algebra provides a beautiful illustration of this, in that it is by nature both algebraic and geometric. Our intuition concerning lines and planes in space acquires an algebraic interpretation that then makes sense more generally in higher dimensions. What's more, in our discussion of the vector space concept, we will see that questions from analysis and differential equations can be approached through linear algebra. Indeed, it is fair to say that linear algebra lies at the foundation of modern mathematics, physics, statistics, and many other disciplines. Linear problems appear in geometry, analysis, and many applied areas. It is this multifaceted aspect of linear algebra that we hope both the instructor and the students will find appealing as they work through this book.

From a pedagogical point of view, linear algebra is an ideal subject for students to learn to think about mathematical concepts and to write rigorous mathematical arguments. One of our goals in writing this text—aside from presenting the standard computational aspects and some interesting applications—is to guide the student in this endeavor. We hope this book will be a thought-provoking introduction to the subject and its myriad applications, one that will be interesting to the science or engineering student but will also help the mathematics student make the transition to more abstract advanced courses.

We have tried to keep the prerequisites for this book to a minimum. Although many of our students will have had a course in multivariable calculus, we do not presuppose any exposure to vectors or vector algebra. We assume only a passing acquaintance with the derivative and integral in Section 6 of Chapter 3 and Section 4 of Chapter 4. Of course, in the discussion of differential equations in Section 3 of Chapter 7, we expect a bit more, including some familiarity with power series, in order for students to understand the matrix exponential.

In the second edition, we have added approximately 20% more examples (a number of which are sample proofs) and exercises—most computational, so that there are now over 210 examples and 545 exercises (many with multiple parts). We have also added solutions to many more exercises at the back of the book, hoping that this will help some of the students; in the case of exercises requiring proofs, these will provide additional worked examples that many students have requested. We continue to believe that good exercises are ultimately what makes a superior mathematics text.

In brief, here are some of the distinctive features of our approach:

- We introduce geometry from the start, using vector algebra to do a bit of analytic geometry in the first section and the dot product in the second.

- We emphasize concepts and understanding *why*, doing proofs in the text and asking the student to do plenty in the exercises. To help the student adjust to a higher level of mathematical rigor, throughout the early portion of the text we provide "blue boxes" discussing matters of logic and proof technique or advice on formulating problem-solving strategies. A complete list of the blue boxes is included at the end of the book for the instructor's and the students' reference.
- We use rotations, reflections, and projections in $\mathbb{R}^2$ as a first brush with the notion of a linear transformation when we introduce matrix multiplication; we then treat linear transformations generally in concert with the discussion of projections. Thus, we motivate the change-of-basis formula by starting with a coordinate system in which a geometrically defined linear transformation is clearly understood and asking for its standard matrix.
- We emphasize orthogonal complements and their role in finding a homogeneous system of linear equations that defines a given subspace of $\mathbb{R}^n$.
- In the last chapter we include topics for the advanced student, such as Jordan canonical form, a classification of the motions of $\mathbb{R}^2$ and $\mathbb{R}^3$, and a discussion of how Mathematica draws two-dimensional images of three-dimensional shapes.

The historical notes at the end of each chapter, prepared with the generous assistance of Paul Lorczak for the first edition, have been left as is. We hope that they give readers an idea how the subject developed and who the key players were.

A few words on miscellaneous symbols that appear in the text: We have marked with an asterisk (*) the problems for which there are answers or solutions at the back of the text. As a guide for the new teacher, we have also marked with a sharp ($\sharp$) those "theoretical" exercises that are important and to which reference is made later. We indicate the end of a proof by the symbol $\square$.

Significant Changes in the Second Edition

- We have added some examples (particularly of proof reasoning) to Chapter 1 and streamlined the discussion in Sections 4 and 5. In particular, we have included a fairly simple proof that the rank of a matrix is well defined and have outlined in an exercise how this simple proof can be extended to show that reduced echelon form is unique. We have also introduced the Leslie matrix and an application to population dynamics in Section 6.
- We have reorganized Chapter 2, adding two new sections: one on linear transformations and one on elementary matrices. This makes our introduction of linear transformations more detailed and more accessible than in the first edition, paving the way for continued exploration in Chapter 4.
- We have combined the sections on linear independence and basis and noticeably streamlined the treatment of the four fundamental subspaces throughout Chapter 3. In particular, we now obtain *all* the orthogonality relations among these four subspaces in Section 2.
- We have altered Section 1 of Chapter 4 somewhat and have completely reorganized the treatment of the change-of-basis theorem. Now we treat first linear maps $T: \mathbb{R}^n \to \mathbb{R}^n$ in Section 3, and we delay to Section 4 the general case and linear maps on abstract vector spaces.
- We have completely reorganized Chapter 5, moving the geometric interpretation of the determinant from Section 1 to Section 3. Until the end of Section 1, we have tied the computation of determinants to row operations only, proving at the end that this implies multilinearity.

- To reiterate, we have added approximately 20% more exercises, most elementary and computational in nature. We have included more solved problems at the back of the book and, in many cases, have added similar new exercises. We have added some additional blue boxes, as well as a table giving the locations of them all. And we have added more examples early in the text, including more sample proof arguments.

Comments on Individual Chapters

We begin in Chapter 1 with a treatment of vectors, first in $\mathbb{R}^2$ and then in higher dimensions, emphasizing the interplay between algebra and geometry. Parametric equations of lines and planes and the notion of linear combination are introduced in the first section, dot products in the second. We next treat systems of linear equations, starting with a discussion of hyperplanes in $\mathbb{R}^n$, then introducing matrices and Gaussian elimination to arrive at reduced echelon form and the parametric representation of the general solution. We then discuss consistency and the relation between solutions of the homogeneous and inhomogeneous systems. We conclude with a selection of applications.

In Chapter 2 we treat the mechanics of matrix algebra, including a first brush with 2×2 matrices as geometrically defined linear transformations. Multiplication of matrices is viewed as a generalization of multiplication of matrices by vectors, introduced in Chapter 1, but then we come to understand that it represents composition of linear transformations. We now have separate sections for inverse matrices and elementary matrices (where the LU decomposition is introduced) and introduce the notion of transpose. We expect that most instructors will treat elementary matrices lightly.

The heart of the traditional linear algebra course enters in Chapter 3, where we deal with subspaces, linear independence, bases, and dimension. Orthogonality is a major theme throughout our discussion, as is the importance of going back and forth between the parametric representation of a subspace of $\mathbb{R}^n$ and its definition as the solution set of a homogeneous system of linear equations. In the fourth section, we officially give the algorithms for constructing bases for the four fundamental subspaces associated to a matrix. In the optional fifth section, we give the interpretation of these fundamental subspaces in the context of graph theory. In the sixth and last section, we discuss various examples of "abstract" vector spaces, concentrating on matrices, polynomials, and function spaces. The Lagrange interpolation formula is derived by defining an appropriate inner product on the vector space of polynomials.

In Chapter 4 we continue with the geometric flavor of the course by discussing projections, least squares solutions of inconsistent systems, and orthogonal bases and the Gram-Schmidt process. We continue our study of linear transformations in the context of the change-of-basis formula. Here we adopt the viewpoint that the matrix of a geometrically defined transformation is often easy to calculate in a coordinate system adapted to the geometry of the situation; then we can calculate its standard matrix by changing coordinates. The diagonalization problem emerges as natural, and we will return to it fully in Chapter 6.

We give a more thorough treatment of determinants in Chapter 5 than is typical for introductory texts. We have, however, moved the geometric interpretation of signed area and signed volume to the last section of the chapter. We characterize the determinant by its behavior under row operations and then give the usual multilinearity properties. In the second section we give the formula for expanding a determinant in cofactors and conclude with Cramer's Rule.

Chapter 6 is devoted to a thorough treatment of eigenvalues, eigenvectors, diagonalizability, and various applications. In the first section we introduce the characteristic polynomial, and in the second we introduce the notions of algebraic and geometric multiplicity and give a sufficient criterion for a matrix with real eigenvalues to be diagonalizable.

In the third section, we solve some difference equations, emphasizing how eigenvalues and eigenvectors give a “normal mode” decomposition of the solution. We conclude the section with an optional discussion of Markov processes and stochastic matrices. In the last section, we prove the Spectral Theorem, which we believe to be—at least in this most basic setting—one of the important theorems all mathematics majors should know; we include a brief discussion of its application to conics and quadric surfaces.

Chapter 7 consists of three independent special topics. In the first section, we discuss the two obstructions that have arisen in Chapter 6 to diagonalizing a matrix—complex eigenvalues and repeated eigenvalues. Although Jordan canonical form does not ordinarily appear in introductory texts, it is conceptually important and widely used in the study of systems of differential equations and dynamical systems. In the second section, we give a brief introduction to the subject of affine transformations and projective geometry, including discussions of the isometries (motions) of $\mathbb{R}^2$ and $\mathbb{R}^3$. We discuss the notion of perspective projection, which is how computer graphics programs draw images on the screen. An amusing theoretical consequence of this discussion is the fact that circles, ellipses, parabolas, and hyperbolas are all “projectively equivalent” (i.e., can all be seen by projecting any one on different viewing screens). The third, and last, section is perhaps the most standard, presenting the matrix exponential and applications to systems of constant-coefficient ordinary differential equations. Once again, eigenvalues and eigenvectors play a central role in “uncoupling” the system and giving rise, physically, to normal modes.

Acknowledgments

We would like to thank our many colleagues and students who’ve suggested improvements to the text. We give special thanks to our colleagues Ed Azoff and Roy Smith, who have suggested improvements for the second edition. Of course, we thank all our students who have endured earlier versions of the text and made suggestions to improve it; we would like to single out Victoria Akin, Paul Iezzi, Alex Russov, and Catherine Taylor for specific contributions. We appreciate the enthusiastic and helpful support of Terri Ward and Katrina Wilhelm at W. H. Freeman. We would also like to thank the following colleagues around the country, who reviewed the manuscript and offered many helpful comments for the improved second edition:

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The authors welcome your comments and suggestions. Please address any e-mail correspondence to shifrin@math.uga.edu or adams@math.uga.edu . And please keep an eye on

http://www.math.uga.edu/~shifrin/LinAlgErrata.pdf

for information on any typos and corrections.

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FOREWORD TO THE INSTRUCTOR

We have provided more material than most (dare we say all?) instructors can comfortably cover in a one-semester course. We believe it is essential to plan the course so as to have time to come to grips with diagonalization and applications of eigenvalues, including at least one day devoted to the Spectral Theorem. Thus, every instructor will have to make choices and elect to treat certain topics lightly, and others not at all. At the end of this Foreword we present a time frame that we tend to follow, but in a standard-length semester with only three hours a week, one must obviously make some choices and some sacrifices. We cannot overemphasize the caveat that one must be careful to move through Chapter 1 in a timely fashion: Even though it is tempting to plumb the depths of every idea in Chapter 1, we believe that spending one-third of the course on Chapters 1 and 2 is sufficient. Don't worry: As you progress, you will revisit and reinforce the basic concepts in the later chapters.

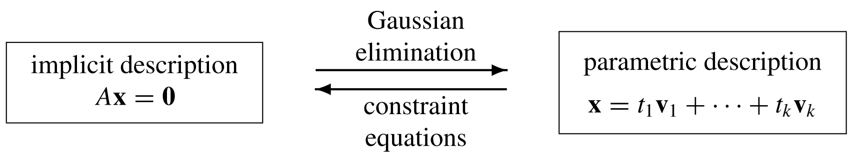
It is also possible to use this text as a second course in linear algebra for students who've had a computational matrix algebra course. For such a course, there should be ample material to cover, treading lightly on the mechanics and spending more time on the theory and various applications, especially Chapter 7.

If you're using this book as your text, we assume that you have a predisposition to teaching proofs and an interest in the geometric emphasis we have tried to provide. We believe strongly that presenting proofs in class is only one ingredient; the students must play an active role by wrestling with proofs in homework as well. To this end, we have provided numerous exercises of varying levels of difficulty that require the students to write proofs. Generally speaking, exercises are arranged in order of increasing difficulty, starting with the computational and ending with the more challenging. To offer a bit more guidance, we have marked with an asterisk (*) those problems for which answers, hints, or detailed proofs are given at the back of the book, and we have marked with a sharp ($^\sharp$) the more theoretical problems that are particularly important (and to which reference is made later). We have added a good number of "asterisked" problems in the second edition. An Instructor's Solutions Manual is available from the publisher.

Although we have parted ways with most modern-day authors of linear algebra textbooks by avoiding technology, we have included a few problems for which a good calculator or computer software will be more than helpful. In addition, when teaching the course, we encourage our students to take advantage of their calculators or available software (e.g., Maple, Mathematica, or MATLAB) to do routine calculations (e.g., reduction to reduced echelon form) once they have mastered the mechanics. Those instructors who are strong believers in the use of technology will no doubt have a preferred supplementary manual to use.

We would like to comment on a few issues that arise when we teach this course.

1. Distinguishing among points in $\mathbb{R}^n$, vectors starting at the origin, and vectors starting elsewhere is always a confusing point at the beginning of any introductory linear algebra text. The rigorous way to deal with this is to define vectors as equivalence classes of ordered pairs of points, but we believe that such an abstract discussion at the outset would be disastrous. Instead, we choose to define vectors to be the "bound" vectors, i.e., the points in the vector space. On the other hand, we use the notion of "free" vector intuitively when discussing geometric notions of vector addition, lines, planes, and the like, because we feel it is essential for our students to develop the geometric intuition that is ubiquitous in physics and geometry.
2. Another mathematical and pedagogical issue is that of using only column vectors to represent elements of $\mathbb{R}^n$. We have chosen to start with the notation $\mathbf{x} = (x_1, \dots, x_n)$ and switch to the column vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ when we introduce matrices in Section 1.4. But for reasons having to do merely with typographical ease, we have not hesitated to use the previous notation from time to time in the text or in exercises when it should cause no confusion.
3. We would encourage instructors using our book for the first time to treat certain topics gently: The material of Section 2.3 is used most prominently in the treatment of determinants. We generally find that it is best to skip the proof of the fundamental Theorem 4.5 in Chapter 3, because we believe that demonstrating it carefully in the case of a well-chosen example is more beneficial to the students. Similarly, we tread lightly in Chapter 5, skipping the proof of Proposition 2.2 in an introductory course. Indeed, when we're pressed for time, we merely remind students of the cofactor expansion in the 3×3 case, prove Cramer's Rule, and move on to Chapter 6. We have moved the discussion of the geometry of determinants to Section 3; instructors who have the extra day or so should certainly include it.
4. To us, one of the big stories in this course is going back and forth between the two ways of describing a subspace $V \subset \mathbb{R}^n$:



   Gaussian elimination gives a basis for the solution space. On the other hand, finding constraint equations that $\mathbf{b}$ must satisfy in order to be a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ gives a system of equations whose solutions are precisely the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.
5. Because we try to emphasize geometry and orthogonality more than most texts, we introduce the orthogonal complement of a subspace early in Chapter 3. In rewriting, we have devoted all of Section 2 to the four fundamental subspaces. We continue to emphasize the significance of the equalities $\mathbf{N}(A) = \mathbf{R}(A)^\perp$ and $\mathbf{N}(A^\top) = \mathbf{C}(A)^\perp$ and the interpretation of the latter in terms of constraint equations. Moreover, we have taken advantage of this interpretation to deduce the companion equalities $\mathbf{C}(A) = \mathbf{N}(A^\top)^\perp$ and $\mathbf{R}(A) = \mathbf{N}(A)^\perp$ immediately, rather than delaying

these as in the first edition. It was confusing enough for the instructor—let alone the poor students—to try to keep track of which we knew and which we didn't. (To deduce $(V^\perp)^\perp = V$ for the *general* subspace $V \subset \mathbb{R}^n$, we need either dimension or the (more basic) fact that every such V has a basis and hence can be expressed as a row or column space.) We hope that our new treatment is both more efficient and less stressful for the students.

6. We always end the course with a proof of the Spectral Theorem and a few days of applications, usually including difference equations and Markov processes (but skipping the optional Section 6.3.1), conics and quadrics, and, if we're lucky, a few days on either differential equations or computer graphics. We do not cover Section 7.1 at all in an introductory course.

7. Instructors who choose to cover abstract vector spaces (Section 3.6) and linear transformations on them (Section 4.4) will discover that most students find this material quite challenging. A few of the exercises will require some calculus skills.

We include the schedule we follow for a one-semester introductory course consisting of forty-five 50-minute class periods, allowing for two or three in-class hour exams. With careful planning, we are able to cover all of the mandatory topics and all of the recommended supplementary topics, but we consider ourselves lucky to have any time at all left for Chapter 7.

Topic	**Recommended Supplementary Topics**	**Sections**	**Days**
Vectors, dot product		1.1–1.2	4
Systems, Gaussian elimination		1.3–1.4	3
Theory of linear systems		1.5	2
	Applications	1.6	2
Matrix algebra, linear maps (treat elementary matrices lightly)		2.1–2.5	6
Vector spaces		3.1–3.4	7
	Abstract vector spaces	3.6	2
Least squares, orthogonal bases		4.1–4.2	3
Change-of-basis formula		4.3	2
	Linear maps on abstract vector spaces	4.4	1
Determinants		5.1–5.2	2.5
	Geometric interpretations	5.3	1
Eigenvalues and eigenvectors		6.1–6.2	3
	Applications	6.3	1.5
Spectral Theorem		6.4	2
		Total:	42

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FOREWORD TO THE STUDENT

We have tried to write a book that you can read—not like a novel, but with pencil in hand. We hope that you will find it interesting, challenging, and rewarding to learn linear algebra. Moreover, by the time you have completed this course, you should find yourself thinking more clearly, analyzing problems with greater maturity, and writing more cogent arguments—both mathematical and otherwise. Above all else, we sincerely hope you will have fun.

To learn mathematics effectively, you must read as an active participant, working through the examples in the text for yourself, *learning all the definitions*, and then attacking lots of exercises—both concrete and theoretical. To this end, there are approximately 550 exercises, a large portion of them having multiple parts. These include computations, applied problems, and problems that ask you to come up with examples. There are proofs varying from the routine to open-ended problems ("Prove or give a counterexample …") to some fairly challenging conceptual posers. It is our intent to help you in your quest to become a better mathematics student. In some cases, studying the examples will provide a direct line of approach to a problem, or perhaps a clue. But in others, you will need to do some independent thinking. Many of the exercises ask you to "prove" or "show" something. To help you learn to think through mathematical problems and write proofs, we've provided 29 "blue boxes" to help you learn basics about the language of mathematics, points of logic, and some pointers on how to approach problem solving and proof writing.

We have provided many examples that demonstrate the ideas and computational tools necessary to do most of the exercises. Nevertheless, you may sometimes believe you have no idea how to get started on a particular problem. Make sure you start by learning the relevant *definitions*. Most of the time in linear algebra, if you know the definition, write down clearly what you are *given*, and note what it is you are to *show*, you are more than halfway there. In a computational problem, before you mechanically write down a matrix and start reducing it to echelon form, be sure you *know* what it is about that matrix that you are trying to find: its row space, its nullspace, its column space, its left nullspace, its eigenvalues, and so on. In more conceptual problems, it may help to make up an example illustrating what you are trying to show; you might try to understand the problem in two or three dimensions—often a picture will give you insight. In other words, learn to play a bit with the problem and feel more comfortable with it. But mathematics can be hard work, and sometimes you should leave a tough problem to "brew" in your brain while you go on to another problem—or perhaps a good night's sleep—to return to it tomorrow.

Remember that in multi-part problems, the hypotheses given at the outset hold throughout the problem. Moreover, *usually* (but not always) we have arranged such problems in such a way that you should use the results of part *a* in trying to do part *b*, and so on. For the problems marked with an asterisk (*) we have provided either numerical answers or, in the case of proof exercises, solutions (some more detailed than others) at the back of the book. Resist as long as possible the temptation to refer to the solutions! Try to be sure you've worked the problem correctly before you glance at the answer. Be careful: Some solutions in the book are not complete, so it is your responsiblity to fill in the details. The problems that are marked with a sharp ($^\sharp$) are not necessarily particularly difficult, but they generally involve concepts and results to which we shall refer later in the text. Thus, if your instructor assigns them, you should make sure you understand how to do them. Occasional exercises are quite challenging, and we hope you will work hard on a few; we firmly believe that only by struggling with a real puzzler do we all progress as mathematicians.

Once again, we hope you will have fun as you embark on your voyage to learn linear algebra. Please let us know if there are parts of the book you find particularly enjoyable or troublesome.

TABLE OF NOTATIONS

Notation	Definition	Page
$\{\ \}$	set	9
$\in$	is an element of	9
$\subset$	is a subset of	12
$\Longrightarrow$	implies	21
$\Longleftrightarrow$	if and only if	21
$\rightsquigarrow$	gives by row operations	41
$\mathbf{A}_i$	i^{th} row vector of the matrix A	39
$\mathbf{a}_j$	j^{th} column vector of the matrix A	53
A^{-1}	inverse of the matrix A	104
A^{T}	transpose of the matrix A	119
A_θ	matrix giving rotation through angle θ	98
$\overline{AB}$	line segment joining A and B	5
$\overrightarrow{AB}$	vector corresponding to the directed line segment from A to B	1
AB	product of the matrices A and B	84
$A\mathbf{x}$	product of the matrix A and the vector $\mathbf{x}$	39
A_{ij}	$(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column from the $n \times n$ matrix A	247
$\mathcal{B}$	basis	227
$\mathbb{C}$	complex numbers	299
$\mathbb{C}^n$	complex n-dimensional space	301
$\mathbf{C}(A)$	column space of the matrix A	136
$\mathcal{C}^k(\mathcal{I})$	vector space of k-times continuously differentiable functions on the interval $\mathcal{I} \subset \mathbb{R}$	178
$\mathcal{C}^\infty(\mathcal{I})$	vector space of infinitely differentiable functions on the interval $\mathcal{I} \subset \mathbb{R}$	178
$C_{\mathcal{B}}$	coordinates with respect to a basis $\mathcal{B}$	227
C_{ij}	ij^{th} cofactor	247
D	differentiation as a linear transformation	225
$D(\mathbf{x}, \mathbf{y})$	signed area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^2$	256
$D(\mathbf{A}_1, \dots, \mathbf{A}_n)$	signed volume of the n-dimensional parallelepiped spanned by $\mathbf{A}_1, \dots, \mathbf{A}_n$	257
$\det A$	determinant of the square matrix A	239
$\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$	standard basis for $\mathbb{R}^n$	149, 213
$\mathbf{E}(\lambda)$	λ-eigenspace	263
e^A	exponential of the square matrix A	333
$\mathcal{F}(\mathcal{I})$	vector space of real-valued functions on the interval $\mathcal{I} \subset \mathbb{R}$	176

Notation	Definition	Page
I_n, I	$n \times n$ identity matrix	87
image (T)	image of a linear transformation T	225
$\ker(T)$	kernel of a linear transformation T	225
$\mathcal{M}_{m\times n}$	vector space of $m \times n$ matrices	82, 176
μ_A	linear transformation defined by multiplication by A	88
$\mathbf{N}(A)$	nullspace of the matrix A	136
$\mathbf{N}(A^{\mathsf{T}})$	left nullspace of the matrix A	138
$\mathcal{P}$	plane, parallelogram, or parallelepiped	11, 255, 258
P_ℓ	projection on a line in $\mathbb{R}^2$	93
P_V	projection on a subspace V	194
$\mathcal{P}$	vector space of polynomials	178
$\mathcal{P}_k$	vector space of polynomials of degree $\le k$	179
$p_A(t)$	characteristic polynomial of the matrix A	265
$\Pi_{\mathbf{a},H}$	projection from $\mathbf{a}$ onto hyperplane H not containing $\mathbf{a}$	323
$\text{proj}_{\mathbf{y}}\mathbf{x}$	projection of $\mathbf{x}$ onto $\mathbf{y}$	22
$\text{proj}_V\mathbf{b}$	projection of $\mathbf{b}$ onto the subspace V	192
$\mathbb{R}$	set of real numbers	1
$\mathbb{R}^2$	Cartesian plane	1
$\mathbb{R}^n$	(real) n-dimensional space	9
$\mathbb{R}^\omega$	vector space of infinite sequences	177
R_ℓ	reflection across a line in $\mathbb{R}^2$	95
R_V	reflection across a subspace V	209
$\mathbf{R}(A)$	row space of the matrix A	138
$\rho(\mathbf{x})$	rotation of $\mathbf{x} \in \mathbb{R}^2$ through angle $\pi/2$	27
Span $(\mathbf{v}_1, \dots, \mathbf{v}_k)$	span of $\mathbf{v}_1, \dots, \mathbf{v}_k$	12
$[T]_{\mathcal{B}}$	matrix of a linear transformation with respect to basis $\mathcal{B}$	212
$[T]_{\text{stand}}$	standard matrix of a linear transformation	209
$[T]_{\mathcal{V},\mathcal{W}}$	matrix of a linear transformation with respect to bases $\mathcal{V}$, $\mathcal{W}$	228
$\text{tr}A$	trace of the matrix A	186
$U + V$	sum of the subspaces U and V	132
$U \cap V$	intersection of the subspaces U and V	135
$\langle \mathbf{u}, \mathbf{v} \rangle$	inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$	181
$\mathbf{u} \times \mathbf{v}$	cross product of the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$	259
$V^\perp$	orthogonal complement of subspace V	133
$\mathcal{V}$, $\mathcal{W}$	ordered bases for vector spaces V and W	228
$\Vert\mathbf{x}\Vert$	length of the vector $\mathbf{x}$	1, 10
$\overline{\mathbf{x}}$	least squares solution	192
$\mathbf{x} \cdot \mathbf{y}$	dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$	19
$\mathbf{x}^\parallel$, $\mathbf{x}^\perp$	components of $\mathbf{x}$ parallel to and orthogonal to another vector	22
$\mathbf{0}$	zero vector	1
O	zero matrix	82

CHAPTER 1

VECTORS AND MATRICES

Linear algebra provides a beautiful example of the interplay between two branches of mathematics: geometry and algebra. We begin this chapter with the geometric concepts and algebraic representations of points, lines, and planes in the more familiar setting of two and three dimensions ($\mathbb{R}^2$ and $\mathbb{R}^3$, respectively) and then generalize to the "n-dimensional" space $\mathbb{R}^n$. We come across two ways of describing (hyper)planes—either parametrically or as solutions of a Cartesian equation. Going back and forth between these two formulations will be a major theme of this text. The fundamental tool that is used in bridging these descriptions is Gaussian elimination, a standard algorithm used to solve systems of linear equations. As we shall see, it also has significant consequences in the theory of systems of equations. We close the chapter with a variety of applications, some not of a geometric nature.

1 Vectors

1.1 Vectors in $\mathbb{R}^2$

Throughout our work the symbol $\mathbb{R}$ denotes the set of real numbers. We define a *vector*[1] in $\mathbb{R}^2$ to be an ordered pair of real numbers, $\mathbf{x} = (x_1, x_2)$. This is the *algebraic* representation of the vector $\mathbf{x}$. Thanks to Descartes, we can identify the ordered pair (x_1, x_2) with a point in the Cartesian plane, $\mathbb{R}^2$. The relationship of this point to the origin $(0, 0)$ gives rise to the *geometric* interpretation of the vector $\mathbf{x}$—namely, the arrow pointing from $(0, 0)$ to (x_1, x_2), as illustrated in Figure 1.1.

The vector $\mathbf{x}$ has *length* and *direction*. The length of $\mathbf{x}$ is denoted $\|\mathbf{x}\|$ and is given by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2},$$

whereas its *direction* can be specified, say, by the angle the arrow makes with the positive x_1-axis. We denote the zero vector $(0, 0)$ by $\mathbf{0}$ and agree that it has no direction. We say two vectors are *equal* if they have the same coordinates, or, equivalently, if they have the same length and direction.

More generally, any two points A and B in the plane determine a directed line segment from A to B, denoted $\overrightarrow{AB}$. This can be visualized as an arrow with A as its "tail" and B as its "head." If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then the arrow $\overrightarrow{AB}$ has the same length

[1]The word derives from the Latin *vector*, "carrier," from *vectus*, the past participle of *vehere*, "to carry."

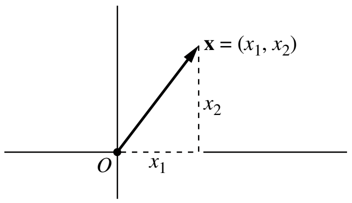


FIGURE 1.1

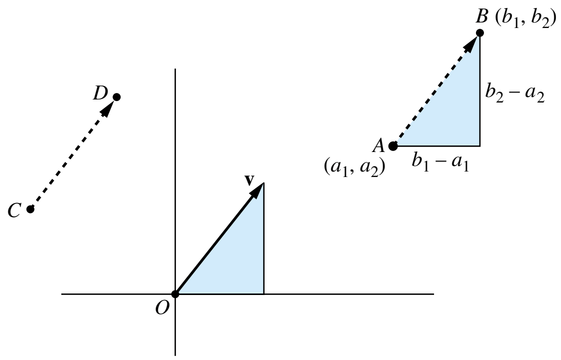


FIGURE 1.2

and direction as the vector $\mathbf{v} = (b_1 - a_1, b_2 - a_2)$. For algebraic purposes, a vector should always have its tail at the origin, but for geometric and physical applications, it is important to be able to "translate" it—to move it parallel to itself so that its tail is elsewhere. Thus, at least geometrically, we think of the arrow $\overrightarrow{AB}$ as the same thing as the vector $\mathbf{v}$. In the same vein, if $C = (c_1, c_2)$ and $D = (d_1, d_2)$, then, as indicated in Figure 1.2, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if $(b_1 - a_1, b_2 - a_2) = (d_1 - c_1, d_2 - c_2)$.[2] This is often a bit confusing at first, so for a while we shall use dotted lines in our diagrams to denote the vectors whose tails are not at the origin.

Scalar multiplication

If c is a real number and $\mathbf{x} = (x_1, x_2)$ is a vector, then we define $c\mathbf{x}$ to be the vector with coordinates (cx_1, cx_2). Now the length of $c\mathbf{x}$ is

$$\|c\mathbf{x}\| = \sqrt{(cx_1)^2 + (cx_2)^2} = \sqrt{c^2(x_1^2 + x_2^2)} = |c|\sqrt{x_1^2 + x_2^2} = |c|\|\mathbf{x}\|.$$

When $c \neq 0$, the direction of $c\mathbf{x}$ is either exactly the same as or exactly opposite that of $\mathbf{x}$, depending on the sign of c. Thus multiplication by the real number c simply stretches (or shrinks) the vector by a factor of $|c|$ and reverses its direction when c is negative, as shown in Figure 1.3. Because this is a geometric "change of scale," we refer to the real number c as a *scalar* and to the multiplication $c\mathbf{x}$ as *scalar multiplication*.

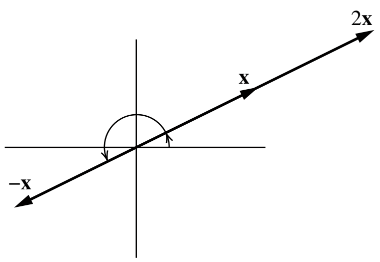


FIGURE 1.3

> ***Definition.*** A vector $\mathbf{x}$ is called a *unit vector* if it has length 1, i.e., if $\|\mathbf{x}\| = 1$.

[2]The sophisticated reader may recognize that we have defined an *equivalence relation* on the collection of directed line segments. A vector can then officially be interpreted as an *equivalence class* of directed line segments.

Note that whenever $\mathbf{x} \neq \mathbf{0}$, we can find a unit vector with the same direction by taking

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\|\mathbf{x}\|}\mathbf{x},$$

as shown in Figure 1.4.

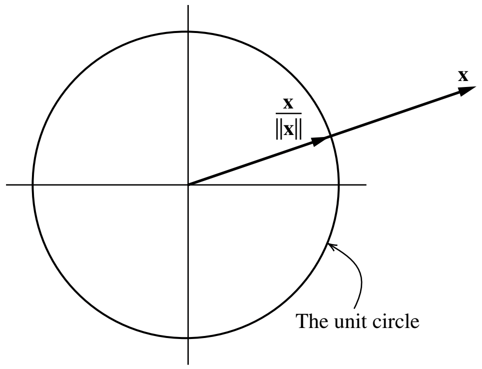


FIGURE 1.4

EXAMPLE 1

The vector $\mathbf{x} = (1, -2)$ has length $\|\mathbf{x}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Thus, the vector

$$\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\sqrt{5}}(1, -2)$$

is a unit vector in the same direction as $\mathbf{x}$. As a check, $\|\mathbf{u}\|^2 = \left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-2}{\sqrt{5}}\right)^2 = \frac{1}{5} + \frac{4}{5} = 1$. ▲

Given a nonzero vector $\mathbf{x}$, any scalar multiple $c\mathbf{x}$ lies on the line that passes through the origin and the head of the vector $\mathbf{x}$. For this reason, we make the following definition.

Definition. We say two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are *parallel* if one vector is a scalar multiple of the other, i.e., if there is a scalar c such that $\mathbf{y} = c\mathbf{x}$. We say two nonzero vectors are *nonparallel* if they are not parallel. (Notice that when one of the vectors is $\mathbf{0}$, they are not considered to be either parallel or nonparallel.)

Vector addition

If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then we define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2).$$

Because addition of real numbers is commutative, it follows immediately that vector addition is commutative:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}.$$

(See Exercise 28 for an exhaustive list of the properties of vector addition and scalar multiplication.) To understand this geometrically, we move the vector **y** so that its tail is at the head of **x** and draw the arrow from the origin to the head of the shifted vector **y**, as shown in Figure 1.5. This is called the *parallelogram law* for vector addition, for, as we see in Figure 1.5, $\mathbf{x}+\mathbf{y}$ is the "long" diagonal of the parallelogram spanned by **x** and **y**. The symmetry of the parallelogram illustrates the commutative law $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$.

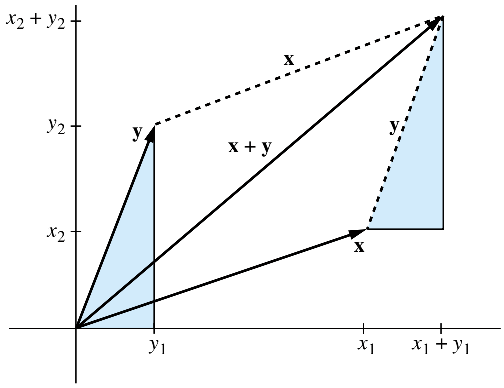


FIGURE 1.5

This would be a good place for the diligent student to grab paper and pencil and make up some numerical examples. Pick a few vectors **x** and **y**, calculate their sums algebraically, and then verify your answers by making sketches to scale.

Remark. We emphasize here that the notions of vector addition and scalar multiplication make sense geometrically for vectors that do not necessarily have their tails at the origin. If we wish to add $\overrightarrow{CD}$ to $\overrightarrow{AB}$, we simply recall that $\overrightarrow{CD}$ is equal to *any* vector with the same length and direction, so we just translate $\overrightarrow{CD}$ so that C and B coincide; then the arrow from A to the point D in its new position is the sum $\overrightarrow{AB}+\overrightarrow{CD}$.

Vector subtraction

Subtraction of one vector from another is also easy to define algebraically. If $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$, then we set

$$\mathbf{x}-\mathbf{y}=(x_1-y_1,x_2-y_2).$$

As is the case with real numbers, we have the following important interpretation of the difference: $\mathbf{x}-\mathbf{y}$ is the vector we must add to **y** in order to obtain **x**; that is,

$$(\mathbf{x}-\mathbf{y})+\mathbf{y}=\mathbf{x}.$$

From this interpretation we can understand $\mathbf{x}-\mathbf{y}$ geometrically. The arrow representing it has its tail at (the head of) **y** and its head at (the head of) **x**; when we add the resulting vector to **y**, we do in fact get **x**. As shown in Figure 1.6, this results in the other diagonal of the parallelogram determined by **x** and **y**. Of course, we can also think of $\mathbf{x}-\mathbf{y}$ as the sum $\mathbf{x}+(-\mathbf{y})=\mathbf{x}+(-1)\mathbf{y}$, as pictured in Figure 1.7. Note that if A and B are points in the plane and O denotes the origin, then setting $\mathbf{x}=\overrightarrow{OB}$ and $\mathbf{y}=\overrightarrow{OA}$ gives $\mathbf{x}-\mathbf{y}=\overrightarrow{AB}$.

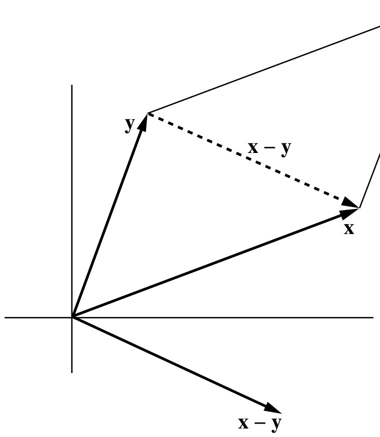


FIGURE 1.6

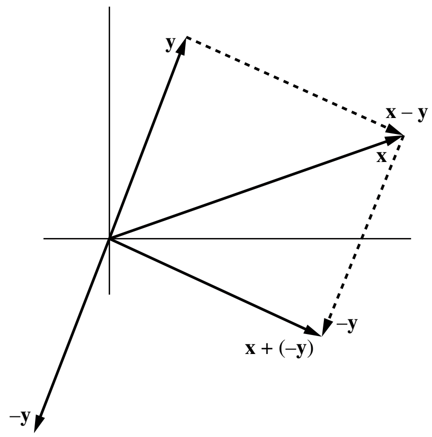


FIGURE 1.7

EXAMPLE 2

Let A and B be points in the plane. The *midpoint* M of the line segment $\overline{AB}$ is the unique point in the plane with the property that $\overrightarrow{AM} = \overrightarrow{MB}$. Since $\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB} = 2\overrightarrow{AM}$, we infer that $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$. (See Figure 1.8.) What's more, we can find the vector $\mathbf{v} = \overrightarrow{OM}$, whose tail is at the origin and whose head is at M, as follows. As above, we set $\mathbf{x} = \overrightarrow{OB}$ and $\mathbf{y} = \overrightarrow{OA}$, so $\overrightarrow{AB} = \mathbf{x} - \mathbf{y}$ and $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{x} - \mathbf{y})$. Then we have

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\ &= \mathbf{y} + \tfrac{1}{2}(\mathbf{x} - \mathbf{y}) \\ &= \mathbf{y} + \tfrac{1}{2}\mathbf{x} - \tfrac{1}{2}\mathbf{y} \\ &= \tfrac{1}{2}\mathbf{x} + \tfrac{1}{2}\mathbf{y} = \tfrac{1}{2}(\mathbf{x} + \mathbf{y}).\end{aligned}$$

In particular, the vector $\overrightarrow{OM}$ is the average of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

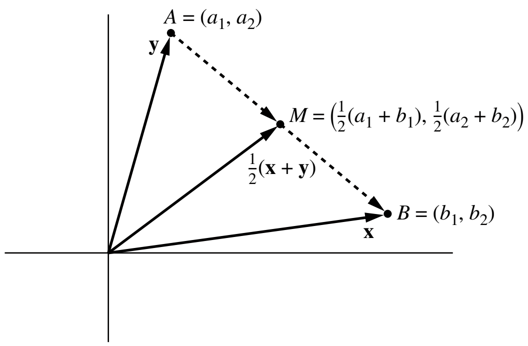


FIGURE 1.8

In coordinates, if $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then the coordinates of M are the average of the respective coordinates of A and B:

$$M = \tfrac{1}{2}\big((a_1, a_2) + (b_1, b_2)\big) = \big(\tfrac{1}{2}(a_1 + b_1), \tfrac{1}{2}(a_2 + b_2)\big).$$

See Exercise 18 for a generalization to three vectors. ▲

We now use the result of Example 2 to derive one of the classic results from high school geometry.

Proposition 1.1. *The diagonals of a parallelogram bisect one another.*

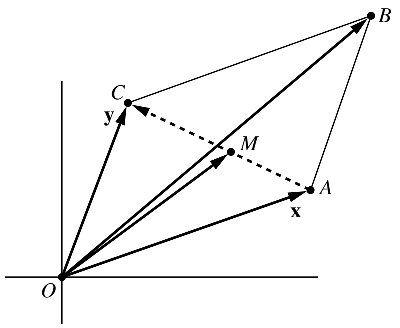


FIGURE 1.9

Proof. The strategy is this: We will find vector expressions for the midpoint of each diagonal and deduce from these expressions that these two midpoints coincide. We may assume one vertex of the parallelogram is at the origin, O, and we label the remaining vertices A, B, and C, as shown in Figure 1.9. Let $\mathbf{x} = \overrightarrow{OA}$ and $\mathbf{y} = \overrightarrow{OC}$, and let M be the midpoint of diagonal $\overline{AC}$. (In the picture, we do not place M on diagonal $\overline{OB}$, even though ultimately we will show that it is on $\overline{OB}$.) We have shown in Example 2 that

$$\overrightarrow{OM} = \tfrac{1}{2}(\mathbf{x} + \mathbf{y}).$$

Next, note that $\overrightarrow{OB} = \mathbf{x} + \mathbf{y}$ by our earlier discussion of vector addition, and so

$$\overrightarrow{ON} = \tfrac{1}{2}\overrightarrow{OB} = \tfrac{1}{2}(\mathbf{x} + \mathbf{y}) = \overrightarrow{OM}.$$

This implies that $M = N$, and so the point M is the midpoint of both diagonals. That is, the two diagonals bisect one another. □

Here is some basic advice in using vectors to prove a geometric statement in $\mathbb{R}^2$. Set up an appropriate diagram and pick two convenient nonparallel vectors that arise naturally in the diagram; call these $\mathbf{x}$ and $\mathbf{y}$, and then express all other relevant quantities in terms of *only* $\mathbf{x}$ and $\mathbf{y}$.

It should now be evident that vector methods provide a great tool for translating theorems from Euclidean geometry into simple algebraic statements. Here is another example. Recall that a *median* of a triangle is a line segment from a vertex to the midpoint of the opposite side.

Proposition 1.2. *The medians of a triangle intersect at a point that is two-thirds of the way from each vertex to the opposite side.*

Proof. We may put one of the vertices of the triangle at the origin, O, so that the picture is as shown at the left in Figure 1.10: Let $\mathbf{x} = \overrightarrow{OA}$, $\mathbf{y} = \overrightarrow{OB}$, and let L, M, and N be the midpoints of $\overline{OA}$, $\overline{AB}$, and $\overline{OB}$, respectively. The battle plan is the following: We let P denote the point two-thirds of the way from B to L, Q the point two-thirds of the way from O to M, and R the point two-thirds of the way from A to N. Although we've indicated P,

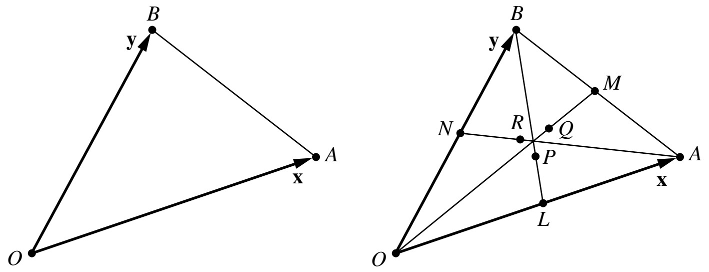


FIGURE 1.10

Q, and R as distinct points at the right in Figure 1.10, our goal is to prove that $P = Q = R$; we do this by expressing all the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ in terms of $\mathbf{x}$ and $\mathbf{y}$. For instance, since $\overrightarrow{OB} = \mathbf{y}$ and $\overrightarrow{OL} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{x}$, we get $\overrightarrow{BL} = \frac{1}{2}\mathbf{x} - \mathbf{y}$, and so

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OB} + \tfrac{2}{3}\overrightarrow{BL} = \mathbf{y} + \tfrac{2}{3}\left(\tfrac{1}{2}\mathbf{x} - \mathbf{y}\right)\\ &= \tfrac{1}{3}\mathbf{x} + \tfrac{1}{3}\mathbf{y}.\end{aligned}$$

Similarly,

$$\begin{aligned}\overrightarrow{OQ} &= \tfrac{2}{3}\overrightarrow{OM} = \tfrac{2}{3}\left(\tfrac{1}{2}(\mathbf{x} + \mathbf{y})\right) = \tfrac{1}{3}(\mathbf{x} + \mathbf{y}); \text{ and}\\ \overrightarrow{OR} &= \overrightarrow{OA} + \overrightarrow{AR} = \overrightarrow{OA} + \tfrac{2}{3}\overrightarrow{AN} = \mathbf{x} + \tfrac{2}{3}\left(\tfrac{1}{2}\mathbf{y} - \mathbf{x}\right) = \tfrac{1}{3}\mathbf{x} + \tfrac{1}{3}\mathbf{y}.\end{aligned}$$

We conclude that, as desired, $\overrightarrow{OP} = \overrightarrow{OQ} = \overrightarrow{OR}$, and so $P = Q = R$. That is, if we go two-thirds of the way down any of the medians, we end up at the same point; this is, of course, the point of intersection of the three medians. □

The astute reader might notice that we could have been more economical in the last proof. Suppose we merely check that the points two-thirds of the way down *two* of the medians (say, P and Q) agree. It would then follow (say, by relabeling the triangle slightly) that the same is true of a different pair of medians (say, P and R). But since any two pairs must have this point in common, we may now conclude that all three points are equal.

1.2 Lines

With these algebraic tools in hand, we now study lines[3] in $\mathbb{R}^2$. A line ℓ_0 through the origin with a given nonzero *direction vector* $\mathbf{v}$ consists of all points of the form $\mathbf{x} = t\mathbf{v}$ for some scalar t. The line ℓ parallel to ℓ_0 and passing through the point P is obtained by translating ℓ_0 by the vector $\mathbf{x}_0 = \overrightarrow{OP}$; that is, the line ℓ through P with direction $\mathbf{v}$ consists of all points of the form

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

as t varies over the real numbers. (It is important to remember that, geometrically, points of the line are the *heads* of the vectors $\mathbf{x}$.) It is compelling to think of t as a time *parameter*; initially (i.e., at time $t = 0$), the point starts at $\mathbf{x}_0$ and moves in the direction of $\mathbf{v}$ as time increases. For this reason, this is often called the *parametric equation* of the line.

To describe the line determined by two distinct points P and Q, we pick $\mathbf{x}_0 = \overrightarrow{OP}$ as before and set $\mathbf{y}_0 = \overrightarrow{OQ}$; we obtain a direction vector by taking

$$\mathbf{v} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{y}_0 - \mathbf{x}_0.$$

[3]Note: In mathematics, the word *line* is reserved for "straight" lines, and the curvy ones are usually called curves.

Thus, as indicated in Figure 1.11, any point on the line through P and Q can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} = \mathbf{x}_0 + t(\mathbf{y}_0 - \mathbf{x}_0) = (1-t)\mathbf{x}_0 + t\mathbf{y}_0.$$

As a check, when $t = 0$ and $t = 1$, we recover the points P and Q, respectively.

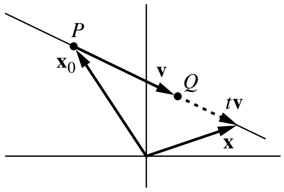


FIGURE 1.11

EXAMPLE 3

Consider the line

$$x_2 = 3x_1 + 1$$

(the usual *Cartesian equation* from high school algebra). We wish to write it in parametric form. Well, any point (x_1, x_2) lying on the line is of the form

$$\mathbf{x} = (x_1, x_2) = (x_1, 3x_1 + 1) = (0, 1) + (x_1, 3x_1) = (0, 1) + x_1(1, 3).$$

Since x_1 can have any real value, we may rename it t, and then, rewriting the equation as

$$\mathbf{x} = (0, 1) + t(1, 3),$$

we recognize this as the equation of the line through the point $P = (0, 1)$ with direction vector $\mathbf{v} = (1, 3)$.

Notice that we might have given alternative parametric equations for this line. The equations

$$\mathbf{x} = (0, 1) + s(2, 6) \quad \text{and} \quad \mathbf{x} = (1, 4) + u(1, 3)$$

also describe this same line. Why? ▲

The "Why?" is a sign that, once again, the reader should take pencil in hand and check that our assertion is correct.

EXAMPLE 4

Consider the line ℓ given in parametric form by

$$\mathbf{x} = (-1, 1) + t(2, 3)$$

and pictured in Figure 1.12. We wish to find a Cartesian equation of the line. Note that ℓ passes through the point $(-1, 1)$ and has direction vector $(2, 3)$. The direction vector determines the slope of the line:

$$\frac{\text{rise}}{\text{run}} = \frac{3}{2},$$

so, using the point-slope form of the equation of a line, we find

$$\frac{x_2 - 1}{x_1 + 1} = \frac{3}{2}; \quad \text{i.e.,} \quad x_2 = \frac{3}{2}x_1 + \frac{5}{2}.$$

Of course, we can rewrite this as $3x_1 - 2x_2 = -5$. ▲

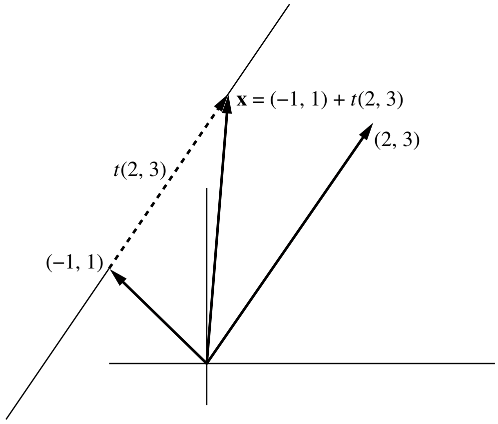


FIGURE 1.12

Mathematics is built around *sets* and relations among them. Although the precise definition of a set is surprisingly subtle, we will adopt the naïve approach that sets are just collections of objects (mathematical or not). The sets with which we shall be concerned in this text consist of vectors. In general, the objects belonging to a set are called its *elements* or *members*. If X is a set and x is an element of X, we write this as

$$x \in X.$$

We might also read the phrase "$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$" as "$\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$" or "$\mathbf{x}$ and $\mathbf{y}$ belong to $\mathbb{R}^n$."

We think of a line in $\mathbb{R}^2$ as the set of points (or vectors) with a certain property. The official notation for the parametric representation is

$$\ell = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (3, 0) + t(-2, 1) \text{ for some scalar } t\}.$$

Or we might describe ℓ by its Cartesian equation:

$$\ell = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + 2x_2 = 3\}.$$

In words, this says that "ℓ is the set of points $\mathbf{x}$ in $\mathbb{R}^2$ such that $x_1 + 2x_2 = 3$."

Often in the text we are sloppy and speak of *the line*

$$(*) \qquad x_1 + 2x_2 = 3$$

rather than using the set notation or saying, more properly, *the line whose equation is* $(*)$.

1.3 On to $\mathbb{R}^n$

The generalizations to $\mathbb{R}^3$ and $\mathbb{R}^n$ are now quite straightforward. A vector $\mathbf{x} \in \mathbb{R}^3$ is defined to be an ordered triple of numbers (x_1, x_2, x_3), which in turn has a geometric interpretation as an arrow from the origin to the point in three-dimensional space with those Cartesian coordinates. Although our geometric intuition becomes hazy when we move to $\mathbb{R}^n$ with $n > 3$, we may still use the algebraic description of a point in n-space as an ordered n-tuple of real numbers $(x_1, x_2, \dots, x_n)$. Thus, we write $\mathbf{x} = (x_1, x_2, \dots, x_n)$ for a vector in n-space. We define $\mathbb{R}^n$ to be the collection of all vectors $(x_1, x_2, \dots, x_n)$ as $x_1, x_2, \dots, x_n$ vary over $\mathbb{R}$. As we did in $\mathbb{R}^2$, given two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, we associate to the directed line segment from A to B the vector $\overrightarrow{AB} = (b_1 - a_1, \dots, b_n - a_n)$.

Remark. The beginning linear algebra student may wonder why anyone would care about $\mathbb{R}^n$ with $n > 3$. We hope that the rich structure we're going to study in this text will eventually be satisfying in and of itself. But some will be happier to know that "real-world applications" force the issue, because many applied problems require understanding the interactions of a large number of variables. For instance, to model the motion of a single particle in $\mathbb{R}^3$, we must know the three variables describing its position *and* the three variables describing its velocity, for a total of six variables. Other examples arise in economic models of a large number of industries, each of which has a supply-demand equation involving large numbers of variables, and in population models describing the interaction of large numbers of different species. In these multivariable problems, each variable accounts for one copy of $\mathbb{R}$, and so an n-variable problem naturally leads to linear (and nonlinear) problems in $\mathbb{R}^n$.

Length, scalar multiplication, and vector addition are defined algebraically in an analogous fashion: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we define

1. $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$;

2. $c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n)$;

3. $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

As before, scalar multiplication stretches (or shrinks or reverses) vectors, and vector addition is given by the parallelogram law. Our notion of length in $\mathbb{R}^n$ is consistent with applying the Pythagorean Theorem (or distance formula); for example, as Figure 1.13 shows, we find the length of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ by first finding the length of the hypotenuse in the x_1x_2-plane and then using that hypotenuse as one leg of the right triangle with hypotenuse $\mathbf{x}$:

$$\|\mathbf{x}\|^2 = \left(\sqrt{x_1^2 + x_2^2}\,\right)^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2.$$

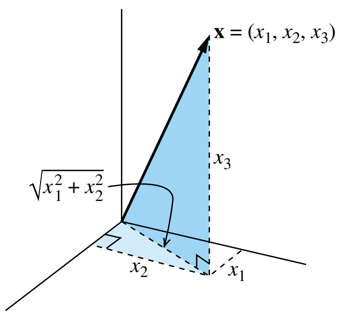


FIGURE 1.13

The *parametric* description of a line ℓ in $\mathbb{R}^n$ is exactly the same as in $\mathbb{R}^2$: If $\mathbf{x}_0 \in \mathbb{R}^n$ is a point on the line and the nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is the direction vector of the line, then points on the line are given by

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, \quad t \in \mathbb{R}.$$

More formally, we write this as

$$\ell = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \text{ for some } t \in \mathbb{R}\}.$$

As we've already seen, two points determine a line; three or more points in $\mathbb{R}^n$ are called *collinear* if they lie on some line; they are called *noncollinear* if they do not lie on any line.

EXAMPLE 5

Consider the line determined by the points $P = (1, 2, 3)$ and $Q = (2, 1, 5)$ in $\mathbb{R}^3$. The direction vector of the line is $\mathbf{v} = \overrightarrow{PQ} = (2, 1, 5) - (1, 2, 3) = (1, -1, 2)$, and we get an initial point $\mathbf{x}_0 = \overrightarrow{OP}$, just as we did in $\mathbb{R}^2$. We now visualize Figure 1.11 as being in $\mathbb{R}^3$ and see that the general point on this line is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} = (1, 2, 3) + t(1, -1, 2)$. ▲

The definition of *parallel* and *nonparallel* vectors in $\mathbb{R}^n$ is identical to that in $\mathbb{R}^2$. Two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ determine a *plane*, $\mathcal{P}_0$, through the origin, as follows. $\mathcal{P}_0$ consists of all points of the form

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

as s and t vary over $\mathbb{R}$. Note that for fixed s, as t varies, the point moves along a line with direction vector $\mathbf{v}$; changing s gives a family of parallel lines. On the other hand, a general plane is determined by one point $\mathbf{x}_0$ and two nonparallel direction vectors $\mathbf{u}$ and $\mathbf{v}$. The plane $\mathcal{P}$ *spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *and passing through the point* $\mathbf{x}_0$ consists of all points $\mathbf{x} \in \mathbb{R}^3$ of the form

$$\mathbf{x} = \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v}$$

as s and t vary over $\mathbb{R}$, as pictured in Figure 1.14. We can obtain the plane $\mathcal{P}$ by translating $\mathcal{P}_0$, the plane parallel to $\mathcal{P}$ and passing through the origin, by the vector $\mathbf{x}_0$. (Note that this *parametric* description of a plane in $\mathbb{R}^3$ makes perfect sense in n-space for any $n \geq 3$.)

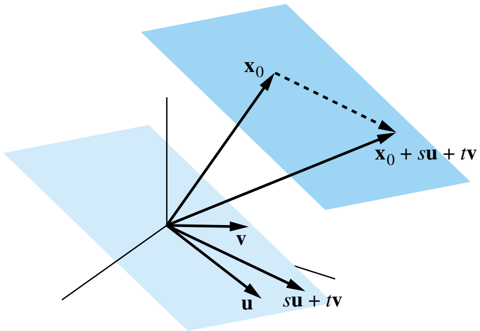


FIGURE 1.14

Before doing some examples, we define two terms that will play a crucial role throughout our study of linear algebra.

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. If $c_1, \ldots, c_k \in \mathbb{R}$, the vector

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is called a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. (See Figure 1.15.)

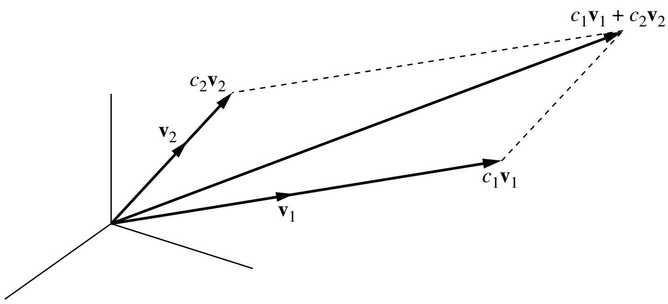


FIGURE 1.15

Definition. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is called their *span*, denoted $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. That is,

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k \text{ for some scalars } c_1, \dots, c_k\}.$$

In terms of our new language, then, the span of two nonparallel vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is a plane through the origin. (What happens if $\mathbf{u}$ and $\mathbf{v}$ are parallel? We will return to such questions in greater generality later in the text.)

EXAMPLE 6

Consider the points $\mathbf{x} \in \mathbb{R}^3$ that satisfy the *Cartesian* equation

$$(\dagger) \qquad x_1 - 2x_2 = 5.$$

The set of points $(x_1, x_2) \in \mathbb{R}^2$ satisfying this equation forms a line ℓ in $\mathbb{R}^2$; since x_3 is allowed to vary arbitrarily, we obtain a vertical plane—a fence standing upon the line ℓ. Let's write it in *parametric* form: Any $\mathbf{x}$ satisfying this equation is of the form

$$\mathbf{x} = (x_1, x_2, x_3) = (5 + 2x_2, x_2, x_3) = (5, 0, 0) + x_2(2, 1, 0) + x_3(0, 0, 1).$$

Since x_2 and x_3 can be arbitrary, we rename them s and t, respectively, obtaining the equation

$$(*) \qquad \mathbf{x} = (5, 0, 0) + s(2, 1, 0) + t(0, 0, 1),$$

which we recognize as a parametric equation of the plane spanned by $(2, 1, 0)$ and $(0, 0, 1)$ and passing through $(5, 0, 0)$. Moreover, note that any $\mathbf{x}$ of this form can be written as $\mathbf{x} = (5 + 2s, s, t)$, and so $x_1 - 2x_2 = (5 + 2s) - 2s = 5$, from which we see that $\mathbf{x}$ is indeed a solution of the equation $(\dagger)$. ▲

This may be an appropriate time to emphasize a basic technique in mathematics: How do we decide when two sets are equal? First of all, we say that X is a *subset* of Y, written

$$X \subset Y,$$

if every element of X is an element of Y. That is, $X \subset Y$ means that whenever $x \in X$, it must also be the case that $x \in Y$. (Some authors write $X \subseteq Y$ to remind us that the sets X and Y may be equal.)

To prove that two sets X and Y are equal (i.e., that every element of X is an element of Y and every element of Y is an element of X), it is often easiest to show that $X \subset Y$ and $Y \subset X$. We ask the diligent reader to check how we've done this explicitly in Example 6: Identify the two sets X and Y, and decide what justifies each of the statements $X \subset Y$ and $Y \subset X$.

EXAMPLE 7

As was the case for lines, a given plane has many different parametric representations. For example,

$$(**) \qquad \mathbf{x} = (7, 1, -5) + u(2, 1, 2) + v(2, 1, 3)$$

is another description of the plane given in Example 6, as we now proceed to check. First, we ask whether every point of $(**)$ can be expressed in the form of $(*)$ for some values of s and t; that is, fixing u and v, we must find s and t so that

$$(5, 0, 0) + s(2, 1, 0) + t(0, 0, 1) = (7, 1, -5) + u(2, 1, 2) + v(2, 1, 3).$$

This gives us the system of equations

$$\begin{aligned} 2s \qquad &= 2u + 2v + 2 \\ s \qquad &= u + v + 1 \\ t &= 2u + 3v - 5, \end{aligned}$$

whose solution is obviously $s = u + v + 1$ and $t = 2u + 3v - 5$. Indeed, we check the algebra:

$$\begin{aligned} (5, 0, 0) + s(2, 1, 0) + t(0, 0, 1) &= (5, 0, 0) + (u + v + 1)(2, 1, 0) \\ &\quad + (2u + 3v - 5)(0, 0, 1) \\ &= \big((5, 0, 0) + (2, 1, 0) - 5(0, 0, 1)\big) \\ &\quad + u\big((2, 1, 0) + 2(0, 0, 1)\big) + v\big((2, 1, 0) + 3(0, 0, 1)\big) \\ &= (7, 1, -5) + u(2, 1, 2) + v(2, 1, 3). \end{aligned}$$

In conclusion, every point of $(**)$ does in fact lie in the plane $(*)$.

Reversing the process is a bit trickier. Given a point of the form $(*)$ for some fixed values of s and t, we need to solve the equations for u and v. We will address this sort of problem in Section 4, but for now, we'll just notice that if we take $u = 3s - t - 8$ and $v = -2s + t + 7$ in the equation $(**)$, we get the point $(*)$. Thus, every point of the plane $(*)$ lies in the plane $(**)$. This means the two planes are, in fact, identical. ▲

EXAMPLE 8

Consider the points $\mathbf{x} \in \mathbb{R}^3$ that satisfy the equation

$$x_1 - 2x_2 + x_3 = 5.$$

Any $\mathbf{x}$ satisfying this equation is of the form

$$\mathbf{x} = (x_1, x_2, x_3) = (5 + 2x_2 - x_3, x_2, x_3) = (5, 0, 0) + x_2(2, 1, 0) + x_3(-1, 0, 1).$$

So this equation describes a plane $\mathcal{P}$ spanned by $(2, 1, 0)$ and $(-1, 0, 1)$ and passing through $(5, 0, 0)$. We leave it to the reader to check the converse—that every point in the plane $\mathcal{P}$ satisfies the original Cartesian equation. ▲

In the preceding examples, we started with a Cartesian equation of a plane in $\mathbb{R}^3$ and derived a parametric formulation. Of course, planes can be described in different ways.

EXAMPLE 9

We wish to find a parametric equation of the plane that contains the points $P = (1, 2, 1)$ and $Q = (2, 4, 0)$ and is parallel to the vector $(1, 1, 3)$. We take $\mathbf{x}_0 = (1, 2, 1)$, $\mathbf{u} = \overrightarrow{PQ} = (1, 2, -1)$, and $\mathbf{v} = (1, 1, 3)$, so the plane consists of all points of the form

$$\mathbf{x} = (1, 2, 1) + s(1, 2, -1) + t(1, 1, 3), \quad s, t \in \mathbb{R}.$$

▲

Finally, note that three noncollinear points $P, Q, R \in \mathbb{R}^3$ determine a plane. To get a parametric equation of this plane, we simply take $\mathbf{x}_0 = \overrightarrow{OP}$, $\mathbf{u} = \overrightarrow{PQ}$, and $\mathbf{v} = \overrightarrow{PR}$. We should observe that if P, Q, and R are noncollinear, then $\mathbf{u}$ and $\mathbf{v}$ are nonparallel (why?).

It is also a reasonable question to ask whether a specific point lies on a given plane.

EXAMPLE 10

Let $\mathbf{u} = (1, 1, 0, -1)$ and $\mathbf{v} = (2, 0, 1, 1)$. We ask whether the vector $\mathbf{x} = (1, 3, -1, -2)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. That is, are there scalars s and t so that $s\mathbf{u} + t\mathbf{v} = \mathbf{x}$, i.e.,

$$s(1, 1, 0, -1) + t(2, 0, 1, 1) = (1, 3, -1, -2)?$$

Expanding, we have

$$(s + 2t, s, t, -s + t) = (1, 3, -1, -2),$$

which leads to the system of equations

$$\begin{aligned} s + 2t &= 1 \\ s &= 3 \\ t &= -1 \\ -s + t &= -2. \end{aligned}$$

From the second and third equations we infer that $s = 3$ and $t = -1$. These values also satisfy the first equation, but *not* the fourth, and so the system of equations has no solution; that is, there are no values of s and t for which *all* the equations hold. Thus, $\mathbf{x}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Geometrically, this means that the vector $\mathbf{x}$ does not lie in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ and passing through the origin. We will learn a systematic way of solving such systems of linear equations in Section 4. ▲

EXAMPLE 11

Suppose that the nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given in $\mathbb{R}^3$ and, moreover, that $\mathbf{v}$ and $\mathbf{w}$ are nonparallel. Consider the line ℓ given parametrically by $\mathbf{x} = \mathbf{x}_0 + r\mathbf{u}$ ($r \in \mathbb{R}$) and the plane $\mathcal{P}$ given parametrically by $\mathbf{x} = \mathbf{x}_1 + s\mathbf{v} + t\mathbf{w}$ ($s, t, \in \mathbb{R}$). Under what conditions do ℓ and $\mathcal{P}$ intersect?

It is a good habit to begin by drawing a sketch to develop some intuition for what the problem is about (see Figure 1.16). We must start by translating the hypothesis that the line and plane have (at least) one point in common into a precise statement involving the parametric equations of the line and plane; our sentence should begin with something like "For some particular values of the real numbers r, s, and t, we have the equation . . . ."

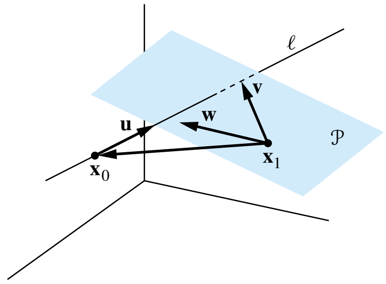


FIGURE 1.16

For ℓ and $\mathcal{P}$ to have (at least) one point $\mathbf{x}^*$ in common, that point must be represented in the form $\mathbf{x}^* = \mathbf{x}_0 + r\mathbf{u}$ for some value of r and, likewise, in the form $\mathbf{x}^* = \mathbf{x}_1 + s\mathbf{v} + t\mathbf{w}$ for some values of s and t. Setting these two expressions for $\mathbf{x}^*$ equal, we have

$$\mathbf{x}_0 + r\mathbf{u} = \mathbf{x}_1 + s\mathbf{v} + t\mathbf{w} \quad \text{for some values of } r, s, \text{ and } t,$$

which holds if and only if

$$\mathbf{x}_0 - \mathbf{x}_1 = -r\mathbf{u} + s\mathbf{v} + t\mathbf{w} \quad \text{for some values of } r, s, \text{ and } t.$$

The latter condition can be rephrased by saying that $\mathbf{x}_0 - \mathbf{x}_1$ lies in Span $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Now, there are two ways this can happen. If Span $(\mathbf{u}, \mathbf{v}, \mathbf{w}) =$ Span $(\mathbf{v}, \mathbf{w})$, then $\mathbf{x}_0 - \mathbf{x}_1$ lies in Span $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ if and only if $\mathbf{x}_0 - \mathbf{x}_1 = s\mathbf{v} + t\mathbf{w}$ for some values of s and t, and this occurs if and only if $\mathbf{x}_0 = \mathbf{x}_1 + s\mathbf{v} + t\mathbf{w}$, i.e., $\mathbf{x}_0 \in \mathcal{P}$. (Geometrically speaking, in this case the line is parallel to the plane, and they intersect if and only if the line is a subset of the plane.) On the other hand, if Span $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$, then ℓ is not parallel to $\mathcal{P}$, and they always intersect. ▲

Exercises 1.1

1. Given $\mathbf{x} = (2, 3)$ and $\mathbf{y} = (-1, 1)$, calculate the following algebraically and sketch a picture to show the geometric interpretation.

a. $\mathbf{x} + \mathbf{y}$ c. $\mathbf{x} + 2\mathbf{y}$ e. $\mathbf{y} - \mathbf{x}$ g. $\|\mathbf{x}\|$

b. $\mathbf{x} - \mathbf{y}$ d. $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ f. $2\mathbf{x} - \mathbf{y}$ h. $\frac{\mathbf{x}}{\|\mathbf{x}\|}$

2. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, compute $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$, and $\mathbf{y} - \mathbf{x}$. Also, provide sketches.

a. $\mathbf{x} = (1, 1)$, $\mathbf{y} = (2, 3)$ c. $\mathbf{x} = (1, 2, -1)$, $\mathbf{y} = (2, 2, 2)$

b. $\mathbf{x} = (2, -2)$, $\mathbf{y} = (0, 2)$

***3.** Three vertices of a parallelogram are $(1, 2, 1)$, $(2, 4, 3)$, and $(3, 1, 5)$. What are all the possible positions of the fourth vertex? Give your reasoning.[4]

4. Let $A = (1, -1, -1)$, $B = (-1, 1, -1)$, $C = (-1, -1, 1)$, and $D = (1, 1, 1)$. Check that the four triangles formed by these points are all equilateral.

***5.** Let ℓ be the line given parametrically by $\mathbf{x} = (1, 3) + t(-2, 1)$, $t \in \mathbb{R}$. Which of the following points lie on ℓ? Give your reasoning.

a. $\mathbf{x} = (-1, 4)$ b. $\mathbf{x} = (7, 0)$ c. $\mathbf{x} = (6, 2)$

[4] For exercises marked with an asterisk (*) we have provided either numerical answers or solutions at the back of the book.

6. Find a parametric equation of each of the following lines:

a. $3x_1 + 4x_2 = 6$

*b. the line with slope $1/3$ that passes through $A = (-1, 2)$

c. the line with slope $2/5$ that passes through $A = (3, 1)$

d. the line through $A = (-2, 1)$ parallel to $\mathbf{x} = (1, 4) + t(3, 5)$

e. the line through $A = (-2, 1)$ perpendicular to $\mathbf{x} = (1, 4) + t(3, 5)$

*f. the line through $A = (1, 2, 1)$ and $B = (2, 1, 0)$

g. the line through $A = (1, -2, 1)$ and $B = (2, 1, -1)$

*h. the line through $(1, 1, 0, -1)$ parallel to $\mathbf{x} = (2 + t, 1 - 2t, 3t, 4 - t)$

7. Suppose $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $\mathbf{y} = \mathbf{y}_0 + s\mathbf{w}$ are two parametric representations of the same line ℓ in $\mathbb{R}^n$.

a. Show that there is a scalar t_0 so that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$.

b. Show that $\mathbf{v}$ and $\mathbf{w}$ are parallel.

***8.** Decide whether each of the following vectors is a linear combination of $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (-2, 1, 0)$.

a. $\mathbf{x} = (1, 0, 0)$ b. $\mathbf{x} = (3, -1, 1)$ c. $\mathbf{x} = (0, 1, 2)$

***9.** Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ spanned by $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (1, -1, 1)$ and passing through the point $(3, 0, -2)$. Which of the following points lie on $\mathcal{P}$?

a. $\mathbf{x} = (4, -1, -1)$

b. $\mathbf{x} = (1, -1, 1)$

c. $\mathbf{x} = (7, -2, 1)$

d. $\mathbf{x} = (5, 2, 0)$

10. Find a parametric equation of each of the following planes:

a. the plane containing the point $(-1, 0, 1)$ and the line $\mathbf{x} = (1, 1, 1) + t(1, 7, -1)$

*b. the plane parallel to the vector $(1, 3, 1)$ and containing the points $(1, 1, 1)$ and $(-2, 1, 2)$

c. the plane containing the points $(1, 1, 2)$, $(2, 3, 4)$, and $(0, -1, 2)$

d. the plane in $\mathbb{R}^4$ containing the points $(1, 1, -1, 2)$, $(2, 3, 0, 1)$, and $(1, 2, 2, 3)$

11. The origin is at the center of a regular m-sided polygon.

a. What is the sum of the vectors from the origin to each of the vertices of the polygon? (The case $m = 7$ is illustrated in Figure 1.17.) Give your reasoning. (*Hint:* What happens if you rotate the vectors by $2\pi/m$?)

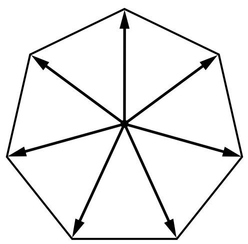


FIGURE 1.17

b. What is the sum of the vectors from one fixed vertex to each of the remaining vertices? (*Hint:* You should use an algebraic approach along with your answer to part *a*.)

***12.** Which of the following are parametric equations of the same plane?

a. $\mathcal{P}_1$: $(1, 1, 0) + s(1, 0, 1) + t(-2, 1, 0)$

b. $\mathcal{P}_2$: $(1, 1, 1) + s(0, 1, 2) + t(2, -1, 0)$

c. $\mathcal{P}_3$: $(2, 0, 0) + s(4, -1, 2) + t(0, 1, 2)$

d. $\mathcal{P}_4$: $(0, 2, 1) + s(1, -1, -1) + t(3, -1, 1)$

13. Given $\triangle ABC$, let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$.

14. Let $ABCD$ be an arbitrary quadrilateral. Let P, Q, R, and S be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Use Exercise 13 to prove that $PQRS$ is a parallelogram.

***15.** In $\triangle ABC$, shown in Figure 1.18, $\|\overrightarrow{AD}\| = \frac{2}{3}\|\overrightarrow{AB}\|$ and $\|\overrightarrow{CE}\| = \frac{2}{5}\|\overrightarrow{CB}\|$. Let Q denote the midpoint of $\overline{CD}$. Show that $\overrightarrow{AQ} = c\overrightarrow{AE}$ for some scalar c, and determine the ratio $c = \|\overrightarrow{AQ}\|/\|\overrightarrow{AE}\|$.

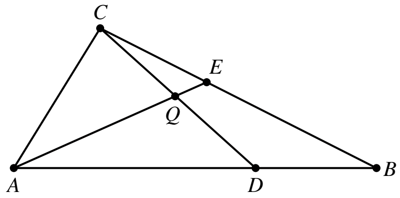


FIGURE 1.18

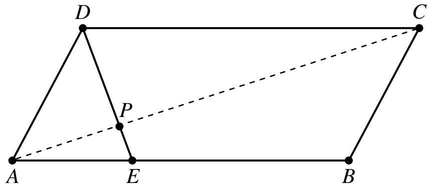


FIGURE 1.19

16. Consider parallelogram $ABCD$. Suppose $\overrightarrow{AE} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{DP} = \frac{3}{4}\overrightarrow{DE}$. Show that P lies on the diagonal $\overline{AC}$. (See Figure 1.19.)

17. Given $\triangle ABC$, suppose that the point D is 3/4 of the way from A to B and that E is the midpoint of $\overline{BC}$. Use vector methods to show that the point P that is 4/7 of the way from C to D is the intersection point of $\overline{CD}$ and $\overline{AE}$.

18. Let A, B, and C be vertices of a triangle in $\mathbb{R}^3$. Let $\mathbf{x} = \overrightarrow{OA}$, $\mathbf{y} = \overrightarrow{OB}$, and $\mathbf{z} = \overrightarrow{OC}$. Show that the head of the vector $\mathbf{v} = \frac{1}{3}(\mathbf{x} + \mathbf{y} + \mathbf{z})$ lies on each median of $\triangle ABC$ (and thus is the point of intersection of the three medians). This point is called the *centroid* of the triangle ABC.

19. a. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Describe the vectors $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, where $s + t = 1$. What particular subset of such $\mathbf{x}$'s is described by $s \geq 0$? By $t \geq 0$? By $s, t > 0$?

b. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Describe the vectors $\mathbf{x} = r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where $r + s + t = 1$. What subsets of such $\mathbf{x}$'s are described by the conditions $r \geq 0$? $s \geq 0$? $t \geq 0$? $r, s, t > 0$?

20. Assume that $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors in $\mathbb{R}^n$. Prove that Span $(\mathbf{u}, \mathbf{v})$ is a line.

21. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scalar. Prove that Span $(\mathbf{v} + c\mathbf{w}, \mathbf{w}) =$ Span $(\mathbf{v}, \mathbf{w})$. (See the blue box on p. 12.)

22. Suppose the vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

a. Prove that for any scalar c, $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

b. Prove that $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

When you are asked to "show" or "prove" something, you should make it a point to write down clearly the information you are *given* and what it is you are *to show*. One word of warning regarding part *b*: To say that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is to say that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$ for *some* scalars $c_1, \dots, c_k$. These scalars will surely be different when you express a different vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, so be sure you give the scalars for $\mathbf{w}$ different names.

***23.** Consider the line ℓ: $\mathbf{x} = \mathbf{x}_0 + r\mathbf{v}$ ($r \in \mathbb{R}$) and the plane $\mathcal{P}$: $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ ($s, t \in \mathbb{R}$). Show that if ℓ and $\mathcal{P}$ intersect, then $\mathbf{x}_0 \in \mathcal{P}$.

24. Consider the lines ℓ: $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $\mathfrak{m}$: $\mathbf{x} = \mathbf{x}_1 + s\mathbf{u}$. Show that ℓ and $\mathfrak{m}$ intersect if and only if $\mathbf{x}_0 - \mathbf{x}_1$ lies in Span $(\mathbf{u}, \mathbf{v})$.

25. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonparallel vectors. (Recall the definition on p. 3.)

a. Prove that if $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$, then $s = t = 0$. (*Hint:* Show that neither $s \neq 0$ nor $t \neq 0$ is possible.)

b. Prove that if $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$, then $a = c$ and $b = d$.

> Two important points emerge in this exercise. First is the appearance of *proof by contradiction*. Although it seems impossible to prove the result of part *a* directly, it is equivalent to prove that if we assume the hypotheses and the *failure* of the conclusion, then we arrive at a contradiction. In this case, if you assume $s\mathbf{x} + t\mathbf{y} = \mathbf{0}$ *and* $s \neq 0$ (or $t \neq 0$), you should be able to see rather easily that $\mathbf{x}$ and $\mathbf{y}$ *are* parallel. In sum, the desired result must be true because it cannot be false.
>
> Next, it is a common (and powerful) technique to prove a result (for example, part *b* of Exercise 25) by first proving a special case (part *a*) and then using it to derive the general case. (Another instance you may have seen in a calculus course is the proof of the Mean Value Theorem by reducing to Rolle's Theorem.)

26. "Discover" the fraction $2/3$ that appears in Proposition 1.2 by finding the intersection of two medians. (Parametrize the line through O and M and the line through A and N, and solve for their point of intersection. You will need to use the result of Exercise 25.)

27. Given $\triangle ABC$, which triangles with vertices on the edges of the original triangle have the same centroid? (See Exercises 18 and 19. At some point, the result of Exercise 25 may be needed, as well.)

28. Verify algebraically that the following properties of vector arithmetic hold. (Do so for $n = 2$ if the general case is too intimidating.) Give the geometric interpretation of each property.

a. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

b. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

c. $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

d. For each $\mathbf{x} \in \mathbb{R}^n$, there is a vector $-\mathbf{x}$ so that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

e. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

f. For all $c \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.

g. For all $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.

h. For all $\mathbf{x} \in \mathbb{R}^n$, $1\mathbf{x} = \mathbf{x}$.

29. a. Using only the properties listed in Exercise 28, prove that for any $\mathbf{x} \in \mathbb{R}^n$, we have $0\mathbf{x} = \mathbf{0}$. (It often surprises students that this is a consequence of the properties in Exercise 28.)

b. Using the result of part *a*, prove that $(-1)\mathbf{x} = -\mathbf{x}$. (Be sure that you didn't use this fact in your proof of part *a*!)

2 Dot Product

We discuss next one of the crucial constructions in linear algebra, the dot product $\mathbf{x} \cdot \mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By way of motivation, let's recall some basic results from plane geometry. Let $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ be points in the plane, as shown in Figure 2.1. We observe that when $\angle POQ$ is a right angle, $\triangle OAP$ is similar to $\triangle OBQ$, and so $x_2/x_1 = -y_1/y_2$, whence $x_1y_1 + x_2y_2 = 0$.

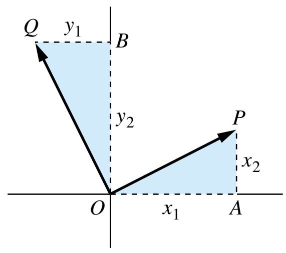


FIGURE 2.1

This leads us to make the following definition.

Definition. Given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, define their *dot product*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2.$$

More generally, given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define their dot product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Remark. The dot product of two vectors is a scalar. For this reason, the dot product is also called the *scalar product*, but it should not be confused with the multiplication of a vector by a scalar, the result of which is a vector. The dot product is also an example of an inner product, which we will study in Section 6 of Chapter 3.

We know that when the vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^2$ are perpendicular, their dot product is 0. By starting with the algebraic properties of the dot product, we are able to get a great deal of geometry out of it.

Proposition 2.1. *The dot product has the following properties:*

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *(the* commutative *property);*
2. $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \geq 0$ *and* $\mathbf{x} \cdot \mathbf{x} = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$*;*
3. $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$*;*
4. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ *for all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *(the* distributive *property).*

Proof. In order to simplify the notation, we give the proof with $n = 2$; the general argument would include all n terms with the obligatory . . . . Because multiplication of real numbers is commutative, we have

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = \mathbf{y} \cdot \mathbf{x}.$$

The square of a real number is nonnegative and the sum of nonnegative numbers is nonnegative, so $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 \geq 0$ and is equal to 0 only when $x_1 = x_2 = 0$.

The next property follows from the associative and distributive properties of real numbers:

$$\begin{aligned}(c\mathbf{x}) \cdot \mathbf{y} &= (cx_1)y_1 + (cx_2)y_2 = c(x_1 y_1) + c(x_2 y_2) \\ &= c(x_1 y_1 + x_2 y_2) = c(\mathbf{x} \cdot \mathbf{y}).\end{aligned}$$

The last result follows from the commutative, associative, and distributive properties of real numbers:

$$\begin{aligned}\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= x_1(y_1 + z_1) + x_2(y_2 + z_2) = x_1 y_1 + x_1 z_1 + x_2 y_2 + x_2 z_2 \\ &= (x_1 y_1 + x_2 y_2) + (x_1 z_1 + x_2 z_2) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}.\end{aligned}$$

□

Corollary 2.2. $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2$.

Proof. Using the properties of Proposition 2.1 repeatedly, we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\\ &= \mathbf{x}\cdot\mathbf{x}+\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{y}\\ &= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2,\end{aligned}$$

as desired. □

Although we use coordinates to define the dot product and to derive its algebraic properties in Proposition 2.1, from this point on we should try to use the properties *themselves* to prove results (e.g., Corollary 2.2). This will tend to avoid an algebraic mess and emphasize the geometry.

The geometric meaning of this result comes from the Pythagorean Theorem: When $\mathbf{x}$ and $\mathbf{y}$ are perpendicular vectors in $\mathbb{R}^2$, as shown in Figure 2.2, we have $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, and so, by Corollary 2.2, it must be the case that $\mathbf{x}\cdot\mathbf{y} = 0$. (And the converse follows, too, from the converse of the Pythagorean Theorem, which follows from the Law of Cosines. See Exercise 14.) That is, two vectors in $\mathbb{R}^2$ are perpendicular if and only if their dot product is 0.

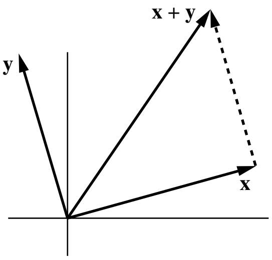


FIGURE 2.2

Motivated by this, we use the algebraic definition of the dot product of vectors in $\mathbb{R}^n$ to bring in the geometry.

Definition. We say vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ are *orthogonal*[5] if $\mathbf{x}\cdot\mathbf{y} = 0$.

Orthogonal and *perpendicular* are synonyms, but we shall stick to the former, because that is the common terminology in linear algebra texts.

EXAMPLE 1

To illustrate the power of the algebraic properties of the dot product, we prove that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus (that is, all sides have equal length). As usual, we place one vertex at the origin (see Figure 2.3),

[5] This word derives from the Greek *orthos*, meaning "straight," "right," or "true," and *gōnia*, meaning "angle."

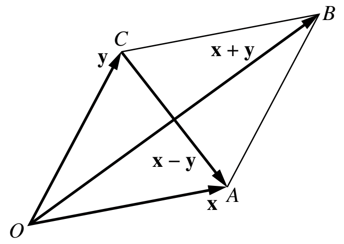


FIGURE 2.3

and we let $\mathbf{x} = \overrightarrow{OA}$ and $\mathbf{y} = \overrightarrow{OC}$ be vectors representing adjacent sides emanating from the origin. We have the diagonals $\overrightarrow{OB} = \mathbf{x} + \mathbf{y}$ and $\overrightarrow{CA} = \mathbf{x} - \mathbf{y}$, so the diagonals are orthogonal if and only if

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = 0.$$

Using the properties of dot product to expand this expression, we obtain

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2,$$

so the diagonals are orthogonal if and only if $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$. Since the length of a vector is nonnegative, this occurs if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$, which means that all the sides of the parallelogram have equal length. ▲

In general, when you are asked to prove a statement of the form P *if and only if* Q, this means that you must prove two statements: If P is true, then Q is also true ("only if"); and if Q is true, then P is also true ("if"). In this example, we gave the two arguments simultaneously, because they relied essentially only on algebraic identities.

A useful shorthand for writing proofs is the *implication* symbol, $\Longrightarrow$. The sentence

$$P \Longrightarrow Q$$

can be read in numerous ways:

- "if P, then Q"
- "P implies Q"
- "P only if Q"
- "Q whenever P"
- "P is sufficient for Q" (because when P is true, then Q is true as well)
- "Q is necessary for P" (because P can't be true unless Q is true)

The "reverse implication" symbol, $\Longleftarrow$, occurs less frequently, because we ordinarily write "$P \Longleftarrow Q$" as "$Q \Longrightarrow P$." This is called the *converse* of the original implication. To convince yourself that a proposition and its converse are logically distinct, consider the sentence "If students major in mathematics, then they take a linear algebra course." The converse is "If students take a linear algebra course, then they major in mathematics." How many of the students in this class are mathematics majors??

We often use the symbol $\Longleftrightarrow$ to denote "if and only if": $P \Longleftrightarrow Q$ means "$P \Longrightarrow Q$ and $Q \Longrightarrow P$." This is often read "P is necessary and sufficient for Q"; here necessity corresponds to "$Q \Longrightarrow P$" and sufficiency corresponds to "$P \Longrightarrow Q$."

Armed with the definition of orthogonal vectors, we proceed to a construction that will be important in much of our future work. Starting with two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\mathbf{y} \neq \mathbf{0}$, Figure 2.4 suggests that we should be able to write $\mathbf{x}$ as the sum of a vector, $\mathbf{x}^{\parallel}$ (read "$\mathbf{x}$-parallel"), that is a scalar multiple of $\mathbf{y}$ and a vector, $\mathbf{x}^{\perp}$ (read "$\mathbf{x}$-perp"), that is orthogonal to $\mathbf{y}$. Let's suppose we have such an equation:

$$\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}, \quad \text{where}$$
$$\mathbf{x}^{\parallel} \text{ is a scalar multiple of } \mathbf{y} \quad \text{and} \quad \mathbf{x}^{\perp} \text{ is orthogonal to } \mathbf{y}.$$

To say that $\mathbf{x}^{\parallel}$ is a scalar multiple of $\mathbf{y}$ means that we can write $\mathbf{x}^{\parallel} = c\mathbf{y}$ for some scalar c. Now, assuming such an expression exists, we can determine c by taking the dot product of both sides of the equation with $\mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x}^{\parallel} + \mathbf{x}^{\perp}) \cdot \mathbf{y} = (\mathbf{x}^{\parallel} \cdot \mathbf{y}) + (\mathbf{x}^{\perp} \cdot \mathbf{y}) = \mathbf{x}^{\parallel} \cdot \mathbf{y} = (c\mathbf{y}) \cdot \mathbf{y} = c\|\mathbf{y}\|^2.$$

This means that

$$c = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}, \quad \text{and so} \quad \mathbf{x}^{\parallel} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

The vector $\mathbf{x}^{\parallel}$ is called the *projection of* $\mathbf{x}$ *onto* $\mathbf{y}$, written $\text{proj}_{\mathbf{y}}\mathbf{x}$.

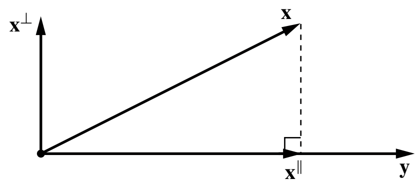


FIGURE 2.4

The fastidious reader may be puzzled by the logic here. We have apparently assumed that we can write $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$ in order to prove that we can do so. Of course, as it stands, this is no fair. Here's how we fix it. We now *define*

$$\mathbf{x}^{\parallel} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}$$
$$\mathbf{x}^{\perp} = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

Obviously, $\mathbf{x}^{\parallel} + \mathbf{x}^{\perp} = \mathbf{x}$ and $\mathbf{x}^{\parallel}$ is a scalar multiple of $\mathbf{y}$. All we need to check is that $\mathbf{x}^{\perp}$ is in fact orthogonal to $\mathbf{y}$. Well,

$$\begin{aligned}
\mathbf{x}^{\perp} \cdot \mathbf{y} &= \left(\mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right) \cdot \mathbf{y} \\
&= \mathbf{x} \cdot \mathbf{y} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \cdot \mathbf{y} \\
&= \mathbf{x} \cdot \mathbf{y} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\|\mathbf{y}\|^2 \\
&= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} = 0,
\end{aligned}$$

as required. Note that by finding a formula for c above, we have shown that $\mathbf{x}^{\parallel}$ is the *unique* multiple of $\mathbf{y}$ that satisfies the equation $(\mathbf{x} - \mathbf{x}^{\parallel}) \cdot \mathbf{y} = 0$.

The pattern of reasoning we've just been through is really not that foreign. When we "solve" the equation

$$\sqrt{x+2} = 2,$$

we assume x satisfies this equation and proceed to find candidates for x. At the end of the process, we must check to see which of our answers work. In this case, of course, we assume x satisfies the equation, square both sides, and conclude that $x = 2$. (That is, *if* $\sqrt{x+2} = 2$, *then* x must equal 2.) But we check the converse: If $x = 2$, then $\sqrt{x+2} = \sqrt{4} = 2$.

It is a bit more interesting if we try solving

$$\sqrt{x+2} = x.$$

Now, squaring both sides leads to the equation

$$x^2 - x - 2 = (x-2)(x+1) = 0,$$

and so we conclude that *if* x satisfies the given equation, *then* $x = 2$ *or* $x = -1$. As before, $x = 2$ is a fine solution, but $x = -1$ is not.

EXAMPLE 2

Let $\mathbf{x} = (2, 3, 1)$ and $\mathbf{y} = (-1, 1, 1)$. Then

$$\mathbf{x}^{\parallel} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \frac{(2,3,1)\cdot(-1,1,1)}{\|(-1,1,1)\|^2}(-1,1,1) = \tfrac{2}{3}(-1,1,1) \quad \text{and}$$
$$\mathbf{x}^{\perp} = (2,3,1) - \tfrac{2}{3}(-1,1,1) = \left(\tfrac{8}{3}, \tfrac{7}{3}, \tfrac{1}{3}\right).$$

To double-check, we compute $\mathbf{x}^{\perp}\cdot\mathbf{y} = \left(\tfrac{8}{3}, \tfrac{7}{3}, \tfrac{1}{3}\right)\cdot(-1,1,1) = 0$, as it should be. ▲

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. We shall see next that the formula for the projection of $\mathbf{x}$ onto $\mathbf{y}$ enables us to calculate the *angle* between the vectors $\mathbf{x}$ and $\mathbf{y}$. Consider the right triangle in Figure 2.5; let θ denote the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. Remembering that the

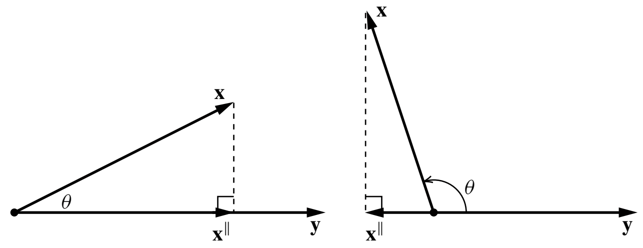


FIGURE 2.5

cosine of an angle is the ratio of the *signed* length of the adjacent side to the length of the hypotenuse, we see that

$$\cos\theta = \frac{\text{signed length of }\mathbf{x}^{\parallel}}{\text{length of }\mathbf{x}} = \frac{c\|\mathbf{y}\|}{\|\mathbf{x}\|} = \frac{\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\|\mathbf{y}\|}{\|\mathbf{x}\|} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

This, then, is the geometric interpretation of the dot product:

$$\boxed{\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta.}$$

Note that if the angle θ is obtuse, i.e., $\pi/2 < |\theta| < \pi$, then $c < 0$ (the signed length of $\mathbf{x}^{\parallel}$ is negative) and $\mathbf{x}\cdot\mathbf{y}$ is negative.

Will this formula still make sense even when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$? Geometrically, we simply restrict our attention to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ and measure the angle θ in that plane, and so we blithely make the following definition.

Definition. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. We define the *angle* between them to be the unique θ satisfying $0 \le \theta \le \pi$ so that

$$\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

EXAMPLE 3

Set $A = (1, -1, -1)$, $B = (-1, 1, -1)$, and $C = (-1, -1, 1)$. Then $\overrightarrow{AB} = (-2, 2, 0)$ and $\overrightarrow{AC} = (-2, 0, 2)$, so

$$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} = \frac{4}{(2\sqrt{2})^2} = \frac{1}{2}.$$

We conclude that $\angle BAC = \pi/3$. ▲

Since our geometric intuition may be misleading in $\mathbb{R}^n$, we should check *algebraically* that this definition makes sense. Since $|\cos\theta| \le 1$, the following result gives us what is needed.

Proposition 2.3 (Cauchy-Schwarz Inequality). *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then*

$$|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Moreover, equality holds if and only if one of the vectors is a scalar multiple of the other.

Proof. If one of the vectors is the zero vector, the result is immediate, so we assume both vectors are nonzero. Suppose first that both $\mathbf{x}$ and $\mathbf{y}$ are unit vectors. Each of the vectors $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ (which we can picture as the diagonals of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$ when the vectors are nonparallel, as shown in Figure 2.6) has nonnegative length.

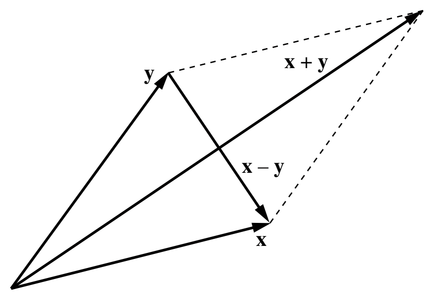


FIGURE 2.6

Using Corollary 2.2, we have

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 = 2(\mathbf{x}\cdot\mathbf{y}+1)$$
$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 = 2(-\mathbf{x}\cdot\mathbf{y}+1).$$

Since $\|\mathbf{x}+\mathbf{y}\|^2 \geq 0$ and $\|\mathbf{x}-\mathbf{y}\|^2 \geq 0$, we see that $\mathbf{x}\cdot\mathbf{y}+1 \geq 0$ and $-\mathbf{x}\cdot\mathbf{y}+1 \geq 0$. Thus,

$$-1 \leq \mathbf{x}\cdot\mathbf{y} \leq 1, \quad \text{and so} \quad |\mathbf{x}\cdot\mathbf{y}| \leq 1.$$

Note that equality holds if and only if either $\mathbf{x}+\mathbf{y}=\mathbf{0}$ or $\mathbf{x}-\mathbf{y}=\mathbf{0}$, i.e., if and only if $\mathbf{x}=\pm\mathbf{y}$.

In general, since $\mathbf{x}/\|\mathbf{x}\|$ and $\mathbf{y}/\|\mathbf{y}\|$ are unit vectors, we have

$$\left|\frac{\mathbf{x}}{\|\mathbf{x}\|}\cdot\frac{\mathbf{y}}{\|\mathbf{y}\|}\right| \leq 1, \quad \text{and so} \quad |\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|,$$

as required. Equality holds if and only if $\dfrac{\mathbf{x}}{\|\mathbf{x}\|} = \pm\dfrac{\mathbf{y}}{\|\mathbf{y}\|}$; that is, equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are parallel. □

Remark. The dot product also arises in situations removed from geometry. The economist introduces the *commodity vector*, whose entries are the quantities of various commodities that happen to be of interest. For example, we might consider $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, where x_1 represents the number of pounds of flour, x_2 the number of dozens of eggs, x_3 the number of pounds of chocolate chips, x_4 the number of pounds of walnuts, and x_5 the number of pounds of butter needed to produce a certain massive quantity of chocolate chip cookies. The economist next introduces the *price vector* $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^5$, where p_i is the price (in dollars) of a unit of the i^{th} commodity (for example, p_2 is the price of a dozen eggs). Then it follows that

$$\mathbf{p}\cdot\mathbf{x} = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + p_5x_5$$

is the total cost of producing the massive quantity of cookies. (To be realistic, we might also want to include x_6 as the number of hours of labor, with corresponding hourly wage p_6.) We will return to this interpretation in Section 5 of Chapter 2.

The gambler uses the dot product to compute the *expected value* of a lottery that has multiple payoffs with various probabilities. If the possible payoffs for a given lottery are given by $\mathbf{w} = (w_1, \dots, w_n)$ and the probabilities of winning the respective payoffs are given by $\mathbf{p} = (p_1, \dots, p_n)$, with $p_1 + \dots + p_n = 1$, then the expected value of the lottery is $\mathbf{p}\cdot\mathbf{w} = p_1w_1 + \dots + p_nw_n$. For example, if the possible prizes, in dollars, for a particular lottery are given by the payoff vector $\mathbf{w} = (0, 1, 5, 100)$ and the probability vector is $\mathbf{p} = (0.5, 0.4, 0.09, 0.01)$, then the expected value is $\mathbf{p}\cdot\mathbf{w} = 0.4 + 0.45 + 1 = 1.85$. Thus, if the lottery ticket costs more than \$1.85, the gambler should expect to lose money in the long run.

Exercises 1.2

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, calculate $\mathbf{x}\cdot\mathbf{y}$ and the angle θ between the vectors.

a. $\mathbf{x} = (2, 5)$, $\mathbf{y} = (-5, 2)$
b. $\mathbf{x} = (2, 1)$, $\mathbf{y} = (-1, 1)$
*c. $\mathbf{x} = (1, 8)$, $\mathbf{y} = (7, -4)$
d. $\mathbf{x} = (1, 4, -3)$, $\mathbf{y} = (5, 1, 3)$
e. $\mathbf{x} = (1, -1, 6)$, $\mathbf{y} = (5, 3, 2)$
*f. $\mathbf{x} = (3, -4, 5)$, $\mathbf{y} = (-1, 0, 1)$
g. $\mathbf{x} = (1, 1, 1, 1)$, $\mathbf{y} = (1, -3, -1, 5)$

*2. For each pair of vectors in Exercise 1, calculate $\text{proj}_{\mathbf{y}}\mathbf{x}$ and $\text{proj}_{\mathbf{x}}\mathbf{y}$.

3. A methane molecule has four hydrogen (H) atoms at the points indicated in Figure 2.7 and a carbon (C) atom at the origin. Find the H − C − H bond angle. (Because of the result of Exercise 1.1.4, this configuration is called a regular tetrahedron.)

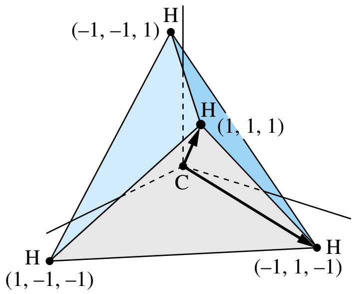


FIGURE 2.7

*4. Find the angle between the long diagonal of a cube and a face diagonal.

5. Find the angle that the long diagonal of a $3 \times 4 \times 5$ rectangular box makes with the longest edge.

*6. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}\| = 3$, $\|\mathbf{y}\| = 2$, and the angle θ between $\mathbf{x}$ and $\mathbf{y}$ is $\theta = \arccos(-1/6)$. Show that the vectors $\mathbf{x} + 2\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal.

7. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}\| = \sqrt{2}$, $\|\mathbf{y}\| = 1$, and the angle between $\mathbf{x}$ and $\mathbf{y}$ is $3\pi/4$. Show that the vectors $2\mathbf{x} + 3\mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal.

8. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ are unit vectors satisfying $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. Determine the angles between each pair of vectors.

9. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ be the so-called *standard basis* for $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{R}^3$ be a nonzero vector. For $i = 1, 2, 3$, let θ_i denote the angle between $\mathbf{x}$ and $\mathbf{e}_i$. Compute $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3$.

*10. Let $\mathbf{x} = (1, 1, 1, \dots, 1) \in \mathbb{R}^n$ and $\mathbf{y} = (1, 2, 3, \dots, n) \in \mathbb{R}^n$. Let θ_n be the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Find $\lim_{n\to\infty} \theta_n$. (The formulas $1 + 2 + \cdots + n = n(n+1)/2$ and $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ may be useful.)

♯11. Suppose $\mathbf{x}, \mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. Show that $\mathbf{x}$ is orthogonal to any linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$.[6]

12. Use vector methods to prove that a parallelogram is a rectangle if and only if its diagonals have the same length.

13. Use the algebraic properties of the dot product to show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right).$$

Interpret the result geometrically.

*14. Use the dot product to prove the law of cosines: As shown in Figure 2.8,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

15. Use vector methods to prove that a triangle that is inscribed in a circle and has a diameter as one of its sides must be a right triangle. (*Hint:* See Figure 2.9. Express the vectors $\mathbf{u}$ and $\mathbf{v}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.)

[6] The symbol ♯ indicates that the result of this problem will be used later.

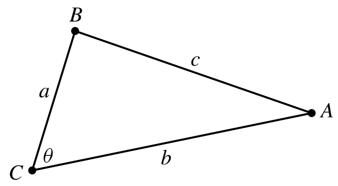


FIGURE 2.8

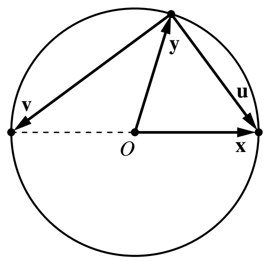


FIGURE 2.9

16. a. Let $\mathbf{y} \in \mathbb{R}^n$. If $\mathbf{x} \cdot \mathbf{y} = 0$ for *all* $\mathbf{x} \in \mathbb{R}^n$, then prove that $\mathbf{y} = \mathbf{0}$.

> When you know some equation holds *for all values of* $\mathbf{x}$, you should often choose some strategic, *particular* value(s) for $\mathbf{x}$.

b. Suppose $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x} \in \mathbb{R}^n$. What can you conclude? (*Hint:* Apply the result of part *a*.)

17. If $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, set $\rho(\mathbf{x}) = (-x_2, x_1)$.

a. Check that $\rho(\mathbf{x})$ is orthogonal to $\mathbf{x}$. (Indeed, $\rho(\mathbf{x})$ is obtained by rotating $\mathbf{x}$ an angle $\pi/2$ counterclockwise.)

b. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, show that $\mathbf{x} \cdot \rho(\mathbf{y}) = -\rho(\mathbf{x}) \cdot \mathbf{y}$. Interpret this statement geometrically.

18. Prove the *triangle inequality*: For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$. (*Hint:* Use the dot product to calculate $\|\mathbf{x} + \mathbf{y}\|^2$.)

19. a. Give an alternative proof of the Cauchy-Schwarz Inequality by minimizing the quadratic function $Q(t) = \|\mathbf{x} - t\mathbf{y}\|^2$. Note that $Q(t) \ge 0$ for all t.

b. If $Q(t_0) \le Q(t)$ for all t, how is $t_0\mathbf{y}$ related to $\mathbf{x}^{\parallel}$? What does this say about $\text{proj}_{\mathbf{y}}\mathbf{x}$?

20. Use the Cauchy-Schwarz inequality to solve the following max/min problem: If the (long) diagonal of a rectangular box has length c, what is the greatest that the sum of the length, width, and height of the box can be? For what shape box does the maximum occur?

21. a. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors with $\|\mathbf{x}\| = \|\mathbf{y}\|$. Prove that the vector $\mathbf{x} + \mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$. (*Hint:* Because $\mathbf{x} + \mathbf{y}$ lies in the plane spanned by $\mathbf{x}$ and $\mathbf{y}$, one has only to check that the angle between $\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ equals the angle between $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$.)

b. More generally, if $\mathbf{x}$ and $\mathbf{y}$ are arbitrary nonzero vectors, let $a = \|\mathbf{x}\|$ and $b = \|\mathbf{y}\|$. Prove that the vector $b\mathbf{x} + a\mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

22. Use vector methods to prove that the diagonals of a parallelogram bisect the vertex angles if and only if the parallelogram is a rhombus. (*Hint:* Use Exercise 21.)

23. Given $\triangle ABC$ with D on $\overline{BC}$, as shown in Figure 2.10, prove that if $\overline{AD}$ bisects $\angle BAC$, then $\|\overrightarrow{BD}\|/\|\overrightarrow{CD}\| = \|\overrightarrow{AB}\|/\|\overrightarrow{AC}\|$. (*Hint:* Use part *b* of Exercise 21. Let $\mathbf{x} = \overrightarrow{AB}$

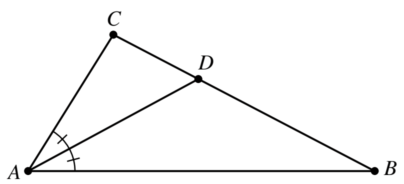


FIGURE 2.10

and $\mathbf{y} = \overrightarrow{AC}$; express $\overrightarrow{AD}$ in two ways as a linear combination of $\mathbf{x}$ and $\mathbf{y}$ and use Exercise 1.1.25.)

24. Use vector methods to show that the angle bisectors of a triangle have a common point. (*Hint:* Given $\triangle OAB$, let $\mathbf{x} = \overrightarrow{OA}$, $\mathbf{y} = \overrightarrow{OB}$, $a = \|\overrightarrow{OA}\|$, $b = \|\overrightarrow{OB}\|$, and $c = \|\overrightarrow{AB}\|$. If we define the point P by $\overrightarrow{OP} = \frac{1}{a+b+c}(b\mathbf{x} + a\mathbf{y})$, use part b of Exercise 21 to show that P lies on all three angle bisectors.)

25. Use vector methods to show that the altitudes of a triangle have a common point. Recall that altitudes of a triangle are the lines passing through a vertex and perpendicular to the line through the remaining vertices. (*Hint:* See Figure 2.11. Let C be the point of intersection of the altitude from B and the altitude from A. Show that $\overrightarrow{OC}$ is orthogonal to $\overrightarrow{AB}$.)

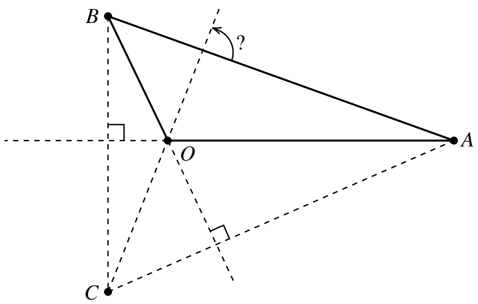


FIGURE 2.11

26. Use vector methods to show that the perpendicular bisectors of the sides of a triangle intersect in a point, as follows. Assume the triangle OAB has one vertex at the origin, and let $\mathbf{x} = \overrightarrow{OA}$ and $\mathbf{y} = \overrightarrow{OB}$. Let $\mathbf{z}$ be the point of intersection of the perpendicular bisectors of $\overline{OA}$ and $\overline{OB}$. Show that $\mathbf{z}$ lies on the perpendicular bisector of $\overline{AB}$. (*Hint:* What is the dot product of $\mathbf{z} - \frac{1}{2}(\mathbf{x} + \mathbf{y})$ with $\mathbf{x} - \mathbf{y}$?)

3 Hyperplanes in $\mathbb{R}^n$

We emphasized earlier a parametric description of lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$. Let's begin by revisiting the Cartesian equation of a line passing through the origin in $\mathbb{R}^2$, e.g.,

$$2x_1 + x_2 = 0.$$

We recognize that the left-hand side of this equation is the dot product of the vector $\mathbf{a} = (2, 1)$ with $\mathbf{x} = (x_1, x_2)$. That is, the vector $\mathbf{x}$ satisfies this equation precisely when it is orthogonal to the vector $\mathbf{a}$, as indicated in Figure 3.1, and we have described the line as the set of vectors in the plane orthogonal to the given vector $\mathbf{a} = (2, 1)$:

$$(*) \qquad \mathbf{a} \cdot \mathbf{x} = 0.$$

It is customary to say that $\mathbf{a}$ is a *normal*[7] *vector* to the line. (Note that any nonzero scalar multiple of $\mathbf{a}$ will do just as well, but we often abuse language by referring to "the" normal vector.)

[7]This is the first of several occurrences of the word *normal*—evidence of mathematicians' propensity to use a word repeatedly with different meanings. Here the meaning derives from the Latin *norma*, "carpenter's square."

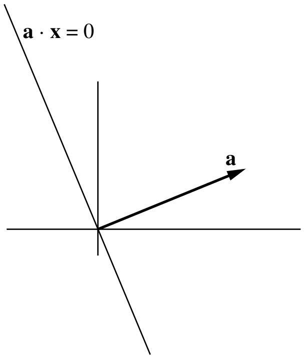


FIGURE 3.1

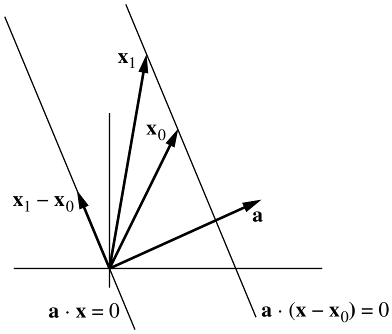


FIGURE 3.2

It is easy to see that specifying a normal vector to a line through the origin is equivalent to specifying its slope. Specifically, if the normal vector is (a, b), then the line has slope $-a/b$. What is the effect of varying the constant on the right-hand side of the equation $(*)$? We get different lines parallel to the one with which we started. In particular, consider a parallel line passing through the point $\mathbf{x}_0$, as shown in Figure 3.2. If $\mathbf{x}$ is on the line, then $\mathbf{x} - \mathbf{x}_0$ will be orthogonal to $\mathbf{a}$, and hence the Cartesian equation of the line is

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0,$$

which we can rewrite in the form

$$\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x}_0$$

or

$$\mathbf{a} \cdot \mathbf{x} = c,$$

where c is the fixed real number $\mathbf{a} \cdot \mathbf{x}_0$.[8] (Why is this quantity the same for every point $\mathbf{x}_0$ on the line?)

EXAMPLE 1

Consider the line ℓ_0 through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{v} = (1, -3)$. The points on this line are all of the form

$$\mathbf{x} = t(1, -3), \quad t \in \mathbb{R}.$$

Because $(3, 1) \cdot (1, -3) = 0$, we may take $\mathbf{a} = (3, 1)$ to be the normal vector to the line, and the Cartesian equation of ℓ_0 is

$$\mathbf{a} \cdot \mathbf{x} = 3x_1 + x_2 = 0.$$

(As a check, suppose we start with $3x_1 + x_2 = 0$. Then we can write $x_1 = -\frac{1}{3}x_2$, and so the solutions consist of vectors of the form

$$\mathbf{x} = (x_1, x_2) = \left(-\tfrac{1}{3}x_2, x_2\right) = -\tfrac{1}{3}x_2(1, -3), \quad x_2 \in \mathbb{R}.$$

Letting $t = -\frac{1}{3}x_2$, we recover the original parametric equation.)

[8] The sophisticated reader should compare this to the study of level curves of functions in multivariable calculus. Here our function is $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$.

Now consider the line ℓ passing through $\mathbf{x}_0 = (2, 1)$ with direction vector $\mathbf{v} = (1, -3)$. Then the points on ℓ are all of the form

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} = (2, 1) + t(1, -3), \quad t \in \mathbb{R}.$$

As promised, we take the same vector $\mathbf{a} = (3, 1)$ and compute that

$$3x_1 + x_2 = \mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot (\mathbf{x}_0 + t\mathbf{v}) = \mathbf{a} \cdot \mathbf{x}_0 + t(\mathbf{a} \cdot \mathbf{v}) = \mathbf{a} \cdot \mathbf{x}_0 = (3, 1) \cdot (2, 1) = 7.$$

This is the Cartesian equation of ℓ. ▲

We can give a geometric interpretation of the constant c on the right-hand side of the equation $\mathbf{a} \cdot \mathbf{x} = c$. Recall that

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a},$$

and so, as indicated in Figure 3.3, the line consists of all vectors whose projection onto the normal vector $\mathbf{a}$ is the constant vector

$$\frac{c}{\|\mathbf{a}\|^2}\mathbf{a}.$$

In particular, since the hypotenuse of a right triangle is longer than either leg,

$$\frac{c}{\|\mathbf{a}\|^2}\mathbf{a}$$

is the point on the line closest to the origin, and we say that the *distance from the origin to the line* is

$$\left\| \frac{c}{\|\mathbf{a}\|^2}\mathbf{a} \right\| = \frac{|c|}{\|\mathbf{a}\|} = \|\text{proj}_{\mathbf{a}}\mathbf{x}_0\|$$

for any point $\mathbf{x}_0$ on the line.

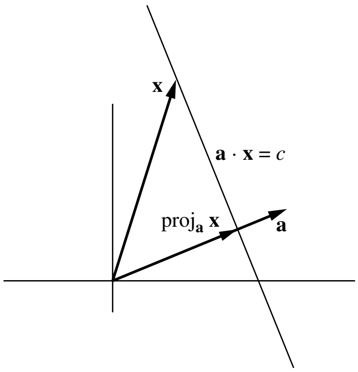


FIGURE 3.3

We now move on to see that planes in $\mathbb{R}^3$ can also be described by using normal vectors.

EXAMPLE 2

Consider the plane $\mathcal{P}_0$ passing through the origin spanned by $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (2, 1, 1)$, as indicated schematically in Figure 3.4. Our intuition suggests that there is a line orthogonal to $\mathcal{P}_0$, so we look for a vector $\mathbf{a} = (a_1, a_2, a_3)$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. It must satisfy the equations

$$\begin{aligned} a_1 \quad\quad + a_3 &= 0 \\ 2a_1 + a_2 + a_3 &= 0. \end{aligned}$$

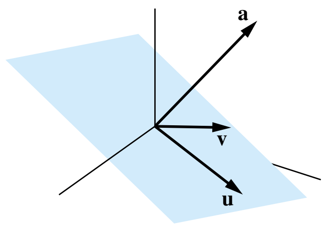


FIGURE 3.4

Substituting $a_3 = -a_1$ into the second equation, we obtain $a_1 + a_2 = 0$, so $a_2 = -a_1$ as well. Thus, any candidate for $\mathbf{a}$ must be a scalar multiple of the vector $(1, -1, -1)$, and so we take $\mathbf{a} = (1, -1, -1)$ and try the equation

$$\mathbf{a} \cdot \mathbf{x} = (1, -1, -1) \cdot \mathbf{x} = x_1 - x_2 - x_3 = 0$$

for $\mathcal{P}_0$. Now, we know that $\mathbf{a} \cdot \mathbf{u} = \mathbf{a} \cdot \mathbf{v} = 0$. Does it follow that $\mathbf{a}$ is orthogonal to every linear combination of $\mathbf{u}$ and $\mathbf{v}$? We just compute: If $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, then

$$\begin{aligned} \mathbf{a} \cdot \mathbf{x} &= \mathbf{a} \cdot (s\mathbf{u} + t\mathbf{v}) \\ &= s(\mathbf{a} \cdot \mathbf{u}) + t(\mathbf{a} \cdot \mathbf{v}) = 0, \end{aligned}$$

as desired.

As before, if we want the equation of the plane $\mathcal{P}$ parallel to $\mathcal{P}_0$ and passing through $\mathbf{x}_0 = (2, 3, -2)$, we take

$$\begin{aligned} x_1 - x_2 - x_3 = \mathbf{a} \cdot \mathbf{x} &= \mathbf{a} \cdot (\mathbf{x}_0 + s\mathbf{u} + t\mathbf{v}) \\ &= \mathbf{a} \cdot \mathbf{x}_0 + s(\mathbf{a} \cdot \mathbf{u}) + t(\mathbf{a} \cdot \mathbf{v}) \\ &= \mathbf{a} \cdot \mathbf{x}_0 = (1, -1, -1) \cdot (2, 3, -2) = 1. \end{aligned}$$ ▲

As this example suggests, a point $\mathbf{x}_0$ and a normal vector $\mathbf{a}$ give rise to the *Cartesian equation* of a plane in $\mathbb{R}^3$:

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0, \quad \text{or, equivalently,} \quad \mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{x}_0.$$

Thus, every plane in $\mathbb{R}^3$ has an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = c,$$

where $\mathbf{a} = (a_1, a_2, a_3)$ is the normal vector and $c \in \mathbb{R}$.

EXAMPLE 3

Consider the set of points $\mathbf{x} = (x_1, x_2, x_3)$ defined by the equation

$$x_1 - 2x_2 + 5x_3 = 3.$$

Let's verify that this is, in fact, a plane in $\mathbb{R}^3$ according to our original parametric definition. If $\mathbf{x}$ satisfies this equation, then $x_1 = 3 + 2x_2 - 5x_3$ and so we may write

$$\begin{aligned} \mathbf{x} = (x_1, x_2, x_3) &= (3 + 2x_2 - 5x_3, x_2, x_3) \\ &= (3, 0, 0) + x_2(2, 1, 0) + x_3(-5, 0, 1). \end{aligned}$$

So, if we let $\mathbf{x}_0 = (3, 0, 0)$, $\mathbf{u} = (2, 1, 0)$, and $\mathbf{v} = (-5, 0, 1)$, we see that $\mathbf{x} = \mathbf{x}_0 + x_2\mathbf{u} + x_3\mathbf{v}$, where x_2 and x_3 are arbitrary scalars. This is in accordance with our original definition of a plane in $\mathbb{R}^3$. ▲

As in the case of lines in $\mathbb{R}^2$, the *distance from the origin to the* (closest point on the) *plane* $\mathbf{a}\cdot\mathbf{x} = c$ is

$$\frac{|c|}{\|\mathbf{a}\|}.$$

Again, note that the point on the plane closest to the origin is

$$\frac{c}{\|\mathbf{a}\|^2}\mathbf{a},$$

which is the point where the line through the origin with direction vector $\mathbf{a}$ intersects the plane, as shown in Figure 3.5. (Indeed, the origin, this point, and any other point $\mathbf{b}$ on the plane form a right triangle, and the hypotenuse of that right triangle has length $\|\mathbf{b}\|$.)

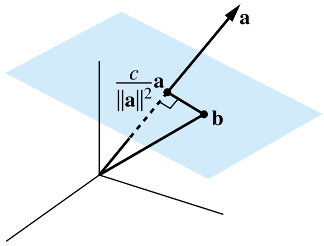


FIGURE 3.5

Finally, generalizing to n dimensions, if $\mathbf{a}\in\mathbb{R}^n$ is a nonzero vector and $c\in\mathbb{R}$, then the equation

$$\mathbf{a}\cdot\mathbf{x} = c$$

defines a *hyperplane* in $\mathbb{R}^n$. As we shall see in Chapter 3, this means that the solution set has "dimension" $n-1$, i.e., 1 less than the dimension of the ambient space $\mathbb{R}^n$. Let's write an explicit formula for the general vector $\mathbf{x}$ satisfying this equation: If $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $a_1 \neq 0$, then we rewrite the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

to solve for x_1:

$$x_1 = \frac{1}{a_1}\left(c - a_2x_2 - \cdots - a_nx_n\right),$$

and so the *general solution* is of the form

$$\begin{aligned}\mathbf{x} = (x_1,\dots,x_n) &= \left(\frac{1}{a_1}\left(c - a_2x_2 - \cdots - a_nx_n\right), x_2, \dots, x_n\right)\\ &= \left(\frac{c}{a_1}, 0, \dots, 0\right) + x_2\left(-\frac{a_2}{a_1}, 1, 0, \dots, 0\right) + x_3\left(-\frac{a_3}{a_1}, 0, 1, \dots, 0\right)\\ &\qquad + \cdots + x_n\left(-\frac{a_n}{a_1}, 0, \dots, 0, 1\right).\end{aligned}$$

(We leave it to the reader to write down the formula in the event that $a_1 = 0$.)

EXAMPLE 4

Consider the hyperplane

$$x_1 + x_2 - x_3 + 2x_4 + x_5 = 2$$

in $\mathbb{R}^5$. Then a parametric description of the general solution of this equation can be written as follows:

$$\begin{aligned}\mathbf{x} &= (-x_2 + x_3 - 2x_4 - x_5 + 2, x_2, x_3, x_4, x_5)\\ &= (2,0,0,0,0) + x_2(-1,1,0,0,0) + x_3(1,0,1,0,0)\\ &\qquad + x_4(-2,0,0,1,0) + x_5(-1,0,0,0,1).\end{aligned}$$

▲

To close this section, let's consider the set of simultaneous solutions of two linear equations in $\mathbb{R}^3$, i.e., the intersection of two planes:

$$\mathbf{a}\cdot\mathbf{x} = a_1x_1 + a_2x_2 + a_3x_3 = c$$
$$\mathbf{b}\cdot\mathbf{x} = b_1x_1 + b_2x_2 + b_3x_3 = d.$$

If a vector $\mathbf{x}$ satisfies both equations, then the point (x_1, x_2, x_3) must lie on both the planes; i.e., it lies in the intersection of the planes. Geometrically, we see that there are three possibilities, as illustrated in Figure 3.6:

1. *A plane*: In this case, both equations describe the same plane.
2. *The empty set*: In this case, the equations describe parallel planes.
3. *A line*: This is the expected situation.

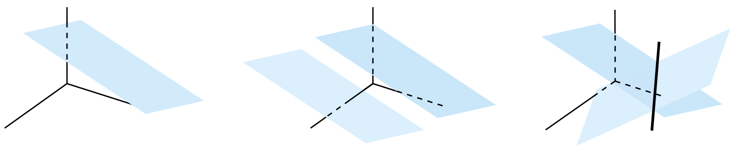


FIGURE 3.6

Notice that if the two planes are identical or parallel, then the normal vectors will be the same (up to a scalar multiple). That is, there will be a nonzero real number r so that $r\mathbf{a} = \mathbf{b}$; if we multiply the equation

$$\mathbf{a}\cdot\mathbf{x} = a_1x_1 + a_2x_2 + a_3x_3 = c$$

by r, we get

$$\mathbf{b}\cdot\mathbf{x} = r\mathbf{a}\cdot\mathbf{x} = b_1x_1 + b_2x_2 + b_3x_3 = rc.$$

If a point (x_1, x_2, x_3) satisfying this equation is also to satisfy the equation

$$\mathbf{b}\cdot\mathbf{x} = b_1x_1 + b_2x_2 + b_3x_3 = d,$$

then we must have $d = rc$; i.e., the two planes coincide. On the other hand, if $d \neq rc$, then there is no solution of the pair of equations, and the two planes are parallel.

More interestingly, if the normal vectors $\mathbf{a}$ and $\mathbf{b}$ are nonparallel, then the planes intersect in a line, and that line is described as the set of solutions of the simultaneous equations. Geometrically, the direction vector of the line must be orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 5

We give a parametric description of the line of intersection of the planes

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_1 - x_2 + 2x_3 &= 5. \end{aligned}$$

Subtracting the first equation from the second yields

$$-3x_2 + 3x_3 = 3, \quad \text{or}$$
$$-x_2 + x_3 = 1.$$

Adding twice the latter equation to the first equation in the original system yields

$$x_1 + x_3 = 4.$$

Thus, we can determine both x_1 and x_2 in terms of x_3:

$$\begin{aligned} x_1 &= 4 - x_3 \\ x_2 &= -1 + x_3 . \end{aligned}$$

Then the general solution is of the form

$$\mathbf{x} = (x_1, x_2, x_3) = (4 - x_3, -1 + x_3, x_3) = (4, -1, 0) + x_3(-1, 1, 1).$$

Indeed, as we mentioned earlier, the direction vector $(-1, 1, 1)$ is orthogonal to $\mathbf{a} = (1, 2, -1)$ and $\mathbf{b} = (1, -1, 2)$. ▲

Much of the remainder of this course will be devoted to understanding higher-dimensional analogues of lines and planes in $\mathbb{R}^3$. In particular, we will be concerned with the relation between their parametric description and their description as the set of solutions of a system of linear equations (geometrically, the intersection of a collection of hyperplanes). The first step toward this goal will be to develop techniques and notation for solving systems of m linear equations in n variables (as in Example 5, where we solved a system of two linear equations in three variables). This is the subject of the next section.

Exercises 1.3

1. Give Cartesian equations of the given hyperplanes:

a. $\mathbf{x} = (-1, 2) + t(3, 2)$

*b. The plane passing through $(1, 2, 2)$ and orthogonal to the line $\mathbf{x} = (5, 1, -1) + t(-1, 1, -1)$

c. The plane passing through $(2, 0, 1)$ and orthogonal to the line $\mathbf{x} = (2, -1, 3) + t(1, 2, -2)$

*d. The plane spanned by $(1, 1, 1)$ and $(2, 1, 0)$ and passing through $(1, 1, 2)$

e. The plane spanned by $(1, 0, 1)$ and $(1, 2, 2)$ and passing through $(-1, 1, 1)$

*f. The hyperplane in $\mathbb{R}^4$ through the origin spanned by $(1, -1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, -1, -1, 1)$.

***2.** Redo Exercise 1.1.12 by finding Cartesian equations of the respective planes.

3. Find the general solution of each of the following equations (presented, as in the text, as a combination of an appropriate number of vectors).

a. $x_1 - 2x_2 + 3x_3 = 4$ (in $\mathbb{R}^3$)

b. $x_1 + x_2 - x_3 + 2x_4 = 0$ (in $\mathbb{R}^4$)

*c. $x_1 + x_2 - x_3 + 2x_4 = 5$ (in $\mathbb{R}^4$)

*d. $x_1 - 2x_2 + 3x_3 = 4$ (in $\mathbb{R}^4$)

e. $x_2 + x_3 - 3x_4 = 2$ (in $\mathbb{R}^4$)

4. Find a normal vector to the given hyperplane and use it to find the distance from the origin to the hyperplane.

a. $\mathbf{x} = (-1, 2) + t(3, 2)$

b. The plane in $\mathbb{R}^3$ given by the equation $2x_1 + x_2 - x_3 = 5$

*c. The plane passing through $(1, 2, 2)$ and orthogonal to the line $\mathbf{x} = (3, 1, -1) + t(-1, 1, -1)$

d. The plane passing through $(2, -1, 1)$ and orthogonal to the line $\mathbf{x} = (3, 1, 1) + t(-1, 2, 1)$

*e. The plane spanned by $(1, 1, 4)$ and $(2, 1, 0)$ and passing through $(1, 1, 2)$

f. The plane spanned by $(1, 1, 1)$ and $(2, 1, 0)$ and passing through $(3, 0, 2)$

g. The hyperplane in $\mathbb{R}^4$ spanned by $(1, -1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, -1, -1, 1)$ and passing through $(2, 1, 0, 1)$

5. Find parametric equations of the line of intersection of the given planes in $\mathbb{R}^3$.

a. $x_1 + x_2 + x_3 = 1$, $2x_1 + x_2 + 2x_3 = 1$

b. $x_1 - x_2 = 1$, $x_1 + x_2 + 2x_3 = 5$

*__6.__ a. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 = 0$ in $\mathbb{R}^3$ (as a linear combination of two vectors, as in the text).

b. Find a specific solution of the equation $x_1 + 5x_2 - 2x_3 = 3$ in $\mathbb{R}^3$; give the general solution.

c. Give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 0$ in $\mathbb{R}^4$. Now give the general solution of the equation $x_1 + 5x_2 - 2x_3 + x_4 = 3$.

*__7.__ The equation $2x_1 - 3x_2 = 5$ defines a line in $\mathbb{R}^2$.

a. Give a normal vector $\mathbf{a}$ to the line.

b. Find the distance from the origin to the line by using projection.

c. Find the point on the line closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your answer to part *b*.

d. Find the distance from the point $\mathbf{w} = (3, 1)$ to the line by using projection.

e. Find the point on the line closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part *d*.

8. The equation $2x_1 - 3x_2 - 6x_3 = -4$ defines a plane in $\mathbb{R}^3$.

a. Give its normal vector $\mathbf{a}$.

b. Find the distance from the origin to the plane by using projection.

c. Find the point on the plane closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your answer to part *b*.

d. Find the distance from the point $\mathbf{w} = (3, -3, -5)$ to the plane by using projection.

e. Find the point on the plane closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part *d*.

9. The equation $2x_1 + 2x_2 - 3x_3 + 8x_4 = 6$ defines a hyperplane in $\mathbb{R}^4$.

a. Give a normal vector $\mathbf{a}$ to the hyperplane.

b. Find the distance from the origin to the hyperplane using projection.

c. Find the point on the hyperplane closest to the origin by using the parametric equation of the line through $\mathbf{0}$ with direction vector $\mathbf{a}$. Double-check your answer to part *b*.

d. Find the distance from the point $\mathbf{w} = (1, 1, 1, 1)$ to the hyperplane using dot products.

e. Find the point on the hyperplane closest to $\mathbf{w}$ by using the parametric equation of the line through $\mathbf{w}$ with direction vector $\mathbf{a}$. Double-check your answer to part *d*.

10. *a. The equations $x_1 = 0$ and $x_2 = 0$ describe planes in $\mathbb{R}^3$ that contain the x_3-axis. Write down the Cartesian equation of a general such plane.

b. The equations $x_1 - x_2 = 0$ and $x_1 - x_3 = 0$ describe planes in $\mathbb{R}^3$ that contain the line through the origin with direction vector $(1, 1, 1)$. Write down the Cartesian equation of a general such plane.

11. a. Assume $\mathbf{b}$ and $\mathbf{c}$ are nonparallel vectors in $\mathbb{R}^3$. Generalizing the result of Exercise 10, show that the plane $\mathbf{a} \cdot \mathbf{x} = 0$ contains the intersection of the planes $\mathbf{b} \cdot \mathbf{x} = 0$ and $\mathbf{c} \cdot \mathbf{x} = 0$ if and only if $\mathbf{a} = s\mathbf{b} + t\mathbf{c}$ for some $s, t \in \mathbb{R}$, not both 0. Describe this result geometrically.

b. Assume $\mathbf{b}$ and $\mathbf{c}$ are nonparallel vectors in $\mathbb{R}^n$. Formulate a conjecture about which hyperplanes $\mathbf{a} \cdot \mathbf{x} = 0$ in $\mathbb{R}^n$ contain the intersection of the hyperplanes $\mathbf{b} \cdot \mathbf{x} = 0$ and $\mathbf{c} \cdot \mathbf{x} = 0$. Prove as much of your conjecture as you can.

12. Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathcal{P} \subset \mathbb{R}^3$ is the plane through the origin with normal vector $\mathbf{a}$. Suppose $\mathcal{P}$ is spanned by $\mathbf{u}$ and $\mathbf{v}$.

a. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathcal{P}$, we have

$$\mathbf{x} = \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}.$$

b. Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. Show that for every $\mathbf{x} \in \mathbb{R}^3$, we have

$$\mathbf{x} = \text{proj}_{\mathbf{a}}\mathbf{x} + \text{proj}_{\mathbf{u}}\mathbf{x} + \text{proj}_{\mathbf{v}}\mathbf{x}.$$

(*Hint:* Apply part *a* to the vector $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}$.)

c. Give an example to show the result of part *a* is false when $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

13. Consider the line ℓ in $\mathbb{R}^3$ given parametrically by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$. Let $\mathcal{P}_0$ denote the plane through the origin with normal vector $\mathbf{a}$ (so it is orthogonal to ℓ).

a. Show that ℓ and $\mathcal{P}_0$ intersect in the point $\mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0$.

b. Conclude that the distance from the origin to ℓ is $\|\mathbf{x}_0 - \text{proj}_{\mathbf{a}}\mathbf{x}_0\|$.

4 Systems of Linear Equations and Gaussian Elimination

In this section we give an explicit algorithm for solving a system of m linear equations in n variables. Unfortunately, this is a little bit like giving the technical description of tying a shoe—it is much easier to do it than to read how to do it. For that reason, before embarking on the technicalities of the process, we will present here a few examples and introduce the notation of matrices. On the other hand, once the technique is mastered, it will be important for us to understand why it yields *all* solutions of the system of equations. For this reason, it is is essential to understand Theorem 4.1.

To begin with, a linear equation in the n variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where the *coefficients* a_i, $i = 1, \dots, n$, are fixed real numbers and b is a fixed real number. Notice that if we let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$, then we can write this equation in vector notation as

$$\mathbf{a} \cdot \mathbf{x} = b.$$

We recognize this as the equation of a hyperplane in $\mathbb{R}^n$, and a vector $\mathbf{x}$ solves the equation precisely when the point $\mathbf{x}$ lies on that hyperplane.

A system of m linear equations in n variables consists of m such equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

The notation appears cumbersome, but we have to live with it. A pair of subscripts is needed on the coefficient a_{ij} to indicate in which equation it appears (the first index, i) and to which variable it is associated (the second index, j). A solution $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of real numbers that satisfies all m of the equations. Thus, a solution gives a point in the intersection of the m hyperplanes.

To *solve* a system of linear equations, we want to give a complete *parametric* description of the solutions, as we did for hyperplanes and for the intersection of two planes in Example 5 in the preceding section. We will call this the *general solution* of the system. Some systems are relatively simple to solve. For example, the system

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 2 \\ x_3 &= -1 \end{aligned}$$

has exactly one solution, namely $\mathbf{x} = (1, 2, -1)$. This is the only point common to the three planes described by the three equations. A slightly more complicated example is

$$\begin{aligned} x_1 \qquad - x_3 &= 1 \\ x_2 + 2x_3 &= 2 . \end{aligned}$$

These equations enable us to determine x_1 and x_2 in terms of x_3; in particular, we can write $x_1 = 1 + x_3$ and $x_2 = 2 - 2x_3$, where x_3 is *free* to take on any real value. Thus, any solution of this system is of the form

$$\mathbf{x} = (1 + t, 2 - 2t, t) = (1, 2, 0) + t(1, -2, 1) \quad \text{for some } t \in \mathbb{R}.$$

It is easily checked that every vector of this form is in fact a solution, as $(1 + t) - t = 1$ and $(2 - 2t) + 2t = 2$ for every $t \in \mathbb{R}$. Thus, we see that the intersection of the two given planes is the line in $\mathbb{R}^3$ passing through $(1, 2, 0)$ with direction vector $(1, -2, 1)$.

One should note that in the preceding example, we chose to solve for x_1 and x_2 in terms of x_3. We could just as well have solved, say, for x_2 and x_3 in terms of x_1 by first writing $x_3 = x_1 - 1$ and then substituting to obtain $x_2 = 4 - 2x_1$. Then we would end up writing

$$\mathbf{x} = (s, 4 - 2s, -1 + s) = (0, 4, -1) + s(1, -2, 1) \quad \text{for some } s \in \mathbb{R}.$$

We will soon give an algorithm for solving systems of linear equations that will eliminate the ambiguity in deciding which variables should be taken as parameters. The variables that are allowed to vary freely (as parameters) are called *free variables*, and the remaining variables, which can be expressed in terms of the free variables, are called *pivot variables*. Broadly speaking, if there are m equations, whenever possible we will try to solve for the first m variables (assuming there are that many) in terms of the remaining variables. This is not always possible (for example, the first variable may not even appear in any of the equations), so we will need to specify a general procedure to select which will be pivot variables and which will be free.

When we are solving a system of equations, there are three basic algebraic operations we can perform that will not affect the solution set. They are the following *elementary operations*:

(i) Interchange any pair of equations.

(ii) Multiply any equation by a nonzero real number.

(iii) Replace any equation by its sum with a multiple of any other equation.

The first two are probably so obvious that it seems silly to write them down; however, soon you will see their importance. It is not obvious that the third operation does not change the solution set; we will address this in Theorem 4.1. First, let's consider an example of solving a system of linear equations using these operations.

EXAMPLE 1

Consider the system of linear equations

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4 \\ 2x_1 + 2x_2 - 2x_3 - 4x_4 &= 6. \end{aligned}$$

We can use operation (i) to replace this system with

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 - 4x_4 &= 6 \\ 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4; \end{aligned}$$

then we use operation (ii), multiplying the first equation by $1/2$, to get

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4; \end{aligned}$$

now we use operation (iii), adding -3 times the first equation to the second:

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ - 5x_2 + 5x_3 + 15x_4 &= -5. \end{aligned}$$

Next we use operation (ii) again, multiplying the second equation by $-1/5$, to obtain

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ x_2 - x_3 - 3x_4 &= 1; \end{aligned}$$

finally, we use operation (iii), adding -1 times the second equation to the first:

$$\begin{aligned} x_1 \qquad\qquad + x_4 &= 2 \\ x_2 - x_3 - 3x_4 &= 1. \end{aligned}$$

From this we see that x_1 and x_2 are determined by x_3 and x_4, both of which are free to take on any values. Thus, we read off the general solution of the system of equations:

$$\begin{aligned} x_1 &= 2 \qquad\quad - x_4 \\ x_2 &= 1 + x_3 + 3x_4 \\ x_3 &= \qquad x_3 \\ x_4 &= \qquad\qquad\quad x_4 \end{aligned}$$

In vector form, the general solution is

$$\mathbf{x} = (x_1, x_2, x_3, x_4) = (2, 1, 0, 0) + x_3(0, 1, 1, 0) + x_4(-1, 3, 0, 1),$$

which is the parametric representation of a plane in $\mathbb{R}^4$. ▲

Before describing the algorithm for solving a general system of linear equations, we want to introduce some notation to make the calculations less cumbersome to write out. We begin with a system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We can simplify our notation somewhat by writing the equations in vector notation:

$$\begin{aligned}\mathbf{A}_1 \cdot \mathbf{x} &= b_1\\ \mathbf{A}_2 \cdot \mathbf{x} &= b_2\\ &\vdots\\ \mathbf{A}_m \cdot \mathbf{x} &= b_m,\end{aligned}$$

where $\mathbf{A}_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{R}^n$, $i = 1, 2, \dots, m$. To simplify the notation further, we introduce the $m \times n$ (read "m by n") *matrix*[9]

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n}\\ a_{21} & \dots & a_{2n}\\ \vdots & \ddots & \vdots\\ a_{m1} & \dots & a_{mn}\end{bmatrix}$$

and the *column vectors*[10]

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix} \in \mathbb{R}^m,$$

and write our equations as

$$A\mathbf{x} = \mathbf{b},$$

where the multiplication on the left-hand side is defined to be

$$A\mathbf{x} = \begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{x}\\ \mathbf{A}_2 \cdot \mathbf{x}\\ \vdots\\ \mathbf{A}_m \cdot \mathbf{x}\end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n\\ a_{21}x_1 + \dots + a_{2n}x_n\\ \vdots\\ a_{m1}x_1 + \dots + a_{mn}x_n\end{bmatrix}.$$

We will discuss the algebraic and geometric properties of matrices a bit later, but for now we simply use them as convenient shorthand notation for systems of equations. We emphasize that an $m \times n$ matrix has m rows and n columns. The coefficient a_{ij} appearing in the i^{th} row and the j^{th} column is called the ij-entry of A. We say that two matrices are *equal* if they have the same *shape* (that is, if they have equal numbers of rows and equal numbers of columns) and their corresponding entries are equal. As we did above, we will customarily denote the *row vectors* of the matrix A by $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{R}^n$.

We reiterate that a solution $\mathbf{x}$ of the system of equations $A\mathbf{x} = \mathbf{b}$ is a vector having the requisite dot products with the row vectors $\mathbf{A}_i$:

$$\mathbf{A}_i \cdot \mathbf{x} = b_i \quad \text{for all } i = 1, 2, \dots, m.$$

That is, the system of equations describes the intersection of the m hyperplanes with normal vectors $\mathbf{A}_i$ and at (signed) distance $b_i/\|\mathbf{A}_i\|$ from the origin. To give the general solution, we must find a parametric representation of this intersection.

---

[9]The word *matrix* derives from the Latin *mātrix*, "womb" (originally, "pregnant animal"), from *māter*, "mother."

[10]We shall henceforth try to write vectors as columns, unless doing so might cause undue typographical hardship.

Notice that the first two types of elementary operations do not change this collection of hyperplanes, so it is no surprise that these operations do not affect the solution set of the system of equations. On the other hand, the third type actually changes one of the hyperplanes without changing the intersection. To see why, suppose $\mathbf{a}$ and $\mathbf{b}$ are nonparallel and consider the pairs of equations

$$\begin{aligned} \mathbf{a}\cdot\mathbf{x} &= 0 \\ \mathbf{b}\cdot\mathbf{x} &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} (\mathbf{a}+c\mathbf{b})\cdot\mathbf{x} &= 0 \\ \mathbf{b}\cdot\mathbf{x} &= 0. \end{aligned}$$

Suppose $\mathbf{x}$ satisfies the first set of equations, so $\mathbf{a}\cdot\mathbf{x}=0$ and $\mathbf{b}\cdot\mathbf{x}=0$; then $\mathbf{x}$ satisfies the second set as well, since $(\mathbf{a}+c\mathbf{b})\cdot\mathbf{x}=(\mathbf{a}\cdot\mathbf{x})+c(\mathbf{b}\cdot\mathbf{x})=0+c0=0$ and $\mathbf{b}\cdot\mathbf{x}=0$ remains true. Conversely, if $\mathbf{x}$ satisfies the second set of equations, we have $\mathbf{b}\cdot\mathbf{x}=0$ and $\mathbf{a}\cdot\mathbf{x}=(\mathbf{a}+c\mathbf{b})\cdot\mathbf{x}-c(\mathbf{b}\cdot\mathbf{x})=0-c0=0$, so $\mathbf{x}$ also satisfies the first set. Thus the solution sets are identical. Geometrically, as shown in Figure 4.1, taking a bit of poetic license, we can think of the hyperplanes $\mathbf{a}\cdot\mathbf{x}=0$ and $\mathbf{b}\cdot\mathbf{x}=0$ as the covers of a book, and the solutions $\mathbf{x}$ will form the "spine" of the book. The typical equation $(\mathbf{a}+c\mathbf{b})\cdot\mathbf{x}=0$ describes one of the pages of the book, and that page intersects either of the covers precisely in the same spine. This follows from the fact that the spine consists of all vectors orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$; this is identical to the plane spanned by $\mathbf{a}+c\mathbf{b}$ and $\mathbf{b}$ (or $\mathbf{a}$).

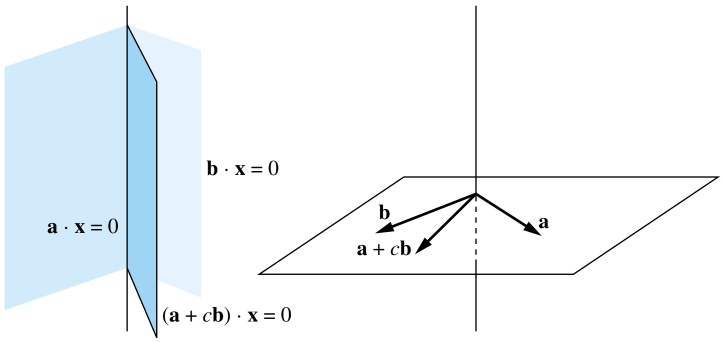


FIGURE 4.1

The general result is the following:

Theorem 4.1. *If a system of equations $A\mathbf{x}=\mathbf{b}$ is changed into the new system $C\mathbf{x}=\mathbf{d}$ by elementary operations, then the systems have the same set of solutions.*

Proof. We need to show that every solution of $A\mathbf{x}=\mathbf{b}$ is also a solution of $C\mathbf{x}=\mathbf{d}$, and vice versa. Start with a solution $\mathbf{u}$ of $A\mathbf{x}=\mathbf{b}$. Denoting the rows of A by $\mathbf{A}_1,\ldots,\mathbf{A}_m$, we have

$$\begin{aligned} \mathbf{A}_1\cdot\mathbf{u} &= b_1 \\ \mathbf{A}_2\cdot\mathbf{u} &= b_2 \\ &\vdots \\ \mathbf{A}_m\cdot\mathbf{u} &= b_m \end{aligned}$$

If we apply an elementary operation of type (i), $\mathbf{u}$ still satisfies precisely the same list of equations. If we apply an elementary operation of type (ii), say multiplying the k^{th} equation by $r\neq 0$, we note that if $\mathbf{u}$ satisfies $\mathbf{A}_k\cdot\mathbf{u}=b_k$, then it must satisfy $(r\mathbf{A}_k)\cdot\mathbf{u}=rb_k$. As for an elementary operation of type (iii), suppose we add r times the k^{th} equation to the ℓ^{th}; since $\mathbf{A}_k \cdot \mathbf{u} = b_k$ and $\mathbf{A}_\ell \cdot \mathbf{u} = b_\ell$, it follows that

$$(r\mathbf{A}_k + \mathbf{A}_\ell) \cdot \mathbf{u} = (r\mathbf{A}_k \cdot \mathbf{u}) + (\mathbf{A}_\ell \cdot \mathbf{u}) = rb_k + b_\ell,$$

and so $\mathbf{u}$ satisfies the "new" ℓ^{th} equation.

To prove conversely that if $\mathbf{u}$ satisfies $C\mathbf{x} = \mathbf{d}$, then it satisfies $A\mathbf{x} = \mathbf{b}$, we merely note that each argument we've given can be reversed; in particular, *the inverse of an elementary operation is again an elementary operation*. Note that it is important here that $r \neq 0$ for an operation of type (ii). □

We introduce one further piece of shorthand notation, the *augmented matrix*

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{cccc|c} a_{11} & \dots & a_{1n} & & b_1 \\ a_{21} & \dots & a_{2n} & & b_2 \\ \vdots & \ddots & \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} & & b_m \end{array}\right].$$

Notice that the augmented matrix contains all of the information of the original system of equations, because we can recover the latter by filling in the x_i's, $+$'s, and $=$'s as needed.

The elementary operations on a system of equations become operations on the rows of the augmented matrix; in this setting, we refer to them as *elementary row operations* of the corresponding three types:

(i) Interchange any pair of rows.

(ii) Multiply all the entries of any row by a nonzero real number.

(iii) Replace any row by its sum with a multiple of any other row.

Since we have established that elementary operations do not affect the solution set of a system of equations, we can freely perform elementary row operations on the augmented matrix of a system of equations with the goal of finding an "equivalent" augmented matrix from which we can easily read off the general solution.

EXAMPLE 2

We revisit Example 1 in the notation of augmented matrices. To solve

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4 \\ 2x_1 + 2x_2 - 2x_3 - 4x_4 &= 6, \end{aligned}$$

we begin by forming the appropriate augmented matrix

$$\left[\begin{array}{cccc|c} 3 & -2 & 2 & 9 & 4 \\ 2 & 2 & -2 & -4 & 6 \end{array}\right].$$

We denote the process of performing row operations by the symbol $\rightsquigarrow$ and (in this example) we indicate above it the type of operation we are performing:

$$\left[\begin{array}{cccc|c} 3 & -2 & 2 & 9 & 4 \\ 2 & 2 & -2 & -4 & 6 \end{array}\right] \overset{\text{(i)}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 2 & 2 & -2 & -4 & 6 \\ 3 & -2 & 2 & 9 & 4 \end{array}\right] \overset{\text{(ii)}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 3 \\ 3 & -2 & 2 & 9 & 4 \end{array}\right]$$

$$\overset{\text{(iii)}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 3 \\ 0 & -5 & 5 & 15 & -5 \end{array}\right] \overset{\text{(ii)}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 3 \\ 0 & 1 & -1 & -3 & 1 \end{array}\right] \overset{\text{(iii)}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 & 1 \end{array}\right].$$

From the final augmented matrix we are able to recover the simpler form of the equations,

$$\begin{aligned} x_1 \qquad\qquad + x_4 &= 2 \\ x_2 - x_3 - 3x_4 &= 1, \end{aligned}$$

and read off the general solution just as before. ▲

Remark. It is important to distinguish between the symbols $=$ and $\rightsquigarrow$; when we convert one matrix to another by performing one or more row operations, we do *not* have equal matrices.

To recap, we have discussed the elementary operations that can be performed on a system of linear equations without changing the solution set, and we have introduced the shorthand notation of augmented matrices. To proceed, we need to discuss the final form our system should have in order for us to be able to read off the solutions easily. To understand this goal, let's consider a few more examples.

EXAMPLE 3

(a) Consider the system

$$\begin{aligned} x_1 + 2x_2 \qquad - x_4 &= 1 \\ x_3 + 2x_4 &= 2. \end{aligned}$$

We see that using the second equation, we can determine x_3 in terms of x_4 and that using the first, we can determine x_1 in terms of x_2 and x_4. In particular, the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-2x_2+x_4 \\ x_2 \\ 2 \qquad -2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

(b) The system

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 3 \\ x_3 + 2x_4 &= 2 \end{aligned}$$

requires some algebraic manipulation before we can read off the solution. Although the second equation determines x_3 in terms of x_4, the first describes x_1 in terms of x_2, x_3, and x_4; but x_2, x_3, and x_4 are not *all* allowed to vary arbitrarily: We would like to modify the first equation by removing x_3. Indeed, if we subtract the second equation from the first, we will recover the system in (a).

(c) The system

$$\begin{aligned} x_1 + 2x_2 \qquad &= 3 \\ x_1 \qquad - x_3 &= 2 \end{aligned}$$

involves similar difficulties. The value of x_1 seems to be determined, on the one hand, by x_2 and, on the other, by x_3; this is problematic (try $x_2 = 1$ and $x_3 = 3$). Indeed, we recognize that this system of equations describes the intersection of two planes in $\mathbb{R}^3$ (that are distinct and not parallel); this should be a line, whose parametric expression should depend on only one variable. The point is that we cannot choose both x_2 and x_3 to be free variables. We first need to manipulate the system of equations so that we can determine one of them in terms of the other (for example, we might subtract the first equation from the second). ▲

The point of this discussion is to use elementary row operations to manipulate systems of linear equations like those in Examples 3(b) and (c) above into equivalent systems from which the solutions can be easily recognized, as in Example 3(a). But what distinguishes Example 3(a)?

Definition. We call the first *nonzero* entry of a row (reading left to right) its *leading entry*. A matrix is in *echelon*[11] *form* if

1. The leading entries move to the right in successive rows.
2. The entries of the column *below* each leading entry are all 0.[12]
3. All rows of 0's are at the bottom of the matrix.

A matrix is in *reduced echelon form* if it is in echelon form and, in addition,

4. Every leading entry is 1.
5. All the entries of the column *above* each leading entry are 0 as well.

If a matrix is in echelon form, we call the leading entry of any (nonzero) row a *pivot*. We refer to the columns in which a pivot appears as *pivot columns* and to the corresponding variables (in the original system of equations) as *pivot variables*. The remaining variables are called *free variables*.

What do we learn from the respective augmented matrices for our earlier examples?

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

Of the augmented matrices from Example 3, (a) is in reduced echelon form, (b) is in echelon form, and (c) is in neither. The key point is this: When the matrix is in reduced echelon form, we are able to determine the general solution by expressing each of the *pivot* variables in terms of the *free* variables.

---

[11] The word *echelon* derives from the French *échelle*, "ladder." Although we don't usually draw the rungs of the ladder, they are there: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. OK, perhaps it looks more like a staircase.

[12] Condition **2** is actually a consequence of **1**, but we state it anyway for clarity.

Here are a few further examples.

EXAMPLE 4

The matrix

$$\begin{bmatrix} 0 & 2 & 1 & 1 & 4 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

is in echelon form. The pivot variables are x_2, x_3, and x_4; the free variables are x_1 and x_5. However, the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not in echelon form, because the row of 0's is not at the bottom; the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

is not in echelon form, since the entry below the leading entry of the second row is nonzero. And the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

in also not in echelon form, because the leading entries do not move to the right. ▲

EXAMPLE 5

The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

is in reduced echelon form. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad + 4x_5 &= 1 \\ x_3 \quad - 2x_5 &= 2 \\ x_4 + x_5 &= 1\,. \end{aligned}$$

Notice that the pivot variables, x_1, x_3, and x_4, are completely determined by the free variables x_2 and x_5. As usual, we can write the general solution in terms of the free variables only:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1-2x_2-4x_5 \\ x_2 \\ 2 \quad +2x_5 \\ 1 \quad - x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

▲

We stop for a moment to formalize the manner in which we have expressed the parametric form of the general solution of a system of linear equations once it's been put in *reduced echelon form*.

Definition. We say that we've written the general solution in *standard form* when it is expressed as the sum of a *particular solution*—obtained by setting all the free variables equal to 0—and a linear combination of vectors, one for each free variable—obtained by setting that free variable equal to 1 and the remaining free variables equal to 0 and ignoring the particular solution.[13]

Our strategy now is to transform the augmented matrix of any system of linear equations into echelon form by performing a sequence of elementary row operations. The algorithm goes by the name of *Gaussian elimination*. The first step is to identify the first column (starting at the left) that does not consist only of 0's; usually this is the first column, but it may not be. Pick a row whose entry in this column is nonzero—usually the uppermost such row, but you may choose another if it helps with the arithmetic—and interchange this with the first row; now the first entry of the first nonzero column is nonzero. This will be our first *pivot*. Next, we add the appropriate multiple of the top row to all the remaining rows to make all the entries below the pivot equal to 0. For example, if we begin with the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 & 7 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & 4 & 2 \end{bmatrix},$$

then we can switch the first and third rows of A (to avoid fractions) and clear out the first pivot column to obtain

$$A' = \begin{bmatrix} \textcircled{2} & 2 & 4 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & -4 & -4 & 4 \end{bmatrix}.$$

We have circled the pivot for emphasis. (If we are headed for the reduced echelon form, we might replace the first row of A' by $(1, 1, 2, 1)$, but this can wait.)

The next step is to find the first column (again, starting at the left) in the *new* matrix having a nonzero entry *below the first row*. Pick a row below the first that has a nonzero entry in this column, and, if necessary, interchange it with the second row. Now the second entry of this column is nonzero; this is our second pivot. (Once again, if we're calculating the reduced echelon form, we multiply by the reciprocal of this entry to make the pivot 1.) We then add appropriate multiples of the second row to the rows beneath it to make all the

[13] In other words, if x_j is a free variable, the corresponding vector in the general solution has j^{th} coordinate equal to 1 and k^{th} coordinate equal to 0 for all the other free variables x_k. Concentrate on the circled entries in the vectors from Example 5:

$$x_2 \begin{bmatrix} -2 \\ \textcircled{1} \\ 0 \\ 0 \\ \textcircled{0} \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ \textcircled{0} \\ 2 \\ -1 \\ \textcircled{1} \end{bmatrix}.$$

entries beneath the pivot equal to 0. Continuing with our example, we obtain

$$A'' = \begin{bmatrix} \textcircled{2} & 2 & 4 & 2 \\ 0 & \textcircled{-1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point, A'' is in echelon form; note that the zero row is at the bottom and that the pivots move toward the right and down.

In general, the process continues until we can find no more pivots—either because we have a pivot in each row or because we're left with nothing but rows of zeroes. At this stage, if we are interested in finding the reduced echelon form, we clear out the entries in the pivot columns *above* the pivots and then make all the pivots equal to 1. (A few words of advice here: If we start at the *right* and work our way up and to the left, we in general minimize the amount of arithmetic that must be done. Also, we always do our best to avoid fractions.) Continuing with our example, we find that the reduced echelon form of A is

$$A'' = \begin{bmatrix} \textcircled{2} & 2 & 4 & 2 \\ 0 & \textcircled{-1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \textcircled{1} & 1 & 2 & 1 \\ 0 & \textcircled{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \textcircled{1} & 0 & 1 & 2 \\ 0 & \textcircled{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_A.$$

It should be evident that there are many choices involved in the process of Gaussian elimination. For example, at the outset, we chose to interchange the first and third rows of A. We might just as well have used either the first or the second row to obtain our first pivot, but we chose the third because we noticed that it would simplify the arithmetic to do so. This lack of specificity in our algorithm is perhaps disconcerting at first, because we are afraid that we might make the "wrong" choice. But so long as we choose a row with a nonzero entry in the appropriate column, we can proceed. It's just a matter of making the arithmetic more or less convenient, and—as in our experience with techniques of integration—practice brings the ability to make more advantageous choices.

Given all the choices we make along the way, we might wonder whether we always arrive at the same answer. Evidently, the echelon form may well depend on the choices. But despite the fact that a matrix may have lots of different echelon forms, they all must have the same number of *nonzero rows*; that number is called the *rank* of the matrix.

Proposition 4.2. *All echelon forms of an $m \times n$ matrix A have the same number of nonzero rows.*

Proof. Suppose B and C are two echelon forms of A, and suppose C has (at least) one more row of zeroes than B. Because there is a pivot in each nonzero row, there is (at least) one pivot variable for B that is a free variable for C, say x_j. Since x_j is a free variable for C, there is a vector $\mathbf{v} = (a_1, a_2, a_3, \dots, a_{j-1}, 1, 0, \dots, 0)$ that satisfies $C\mathbf{v} = \mathbf{0}$. We obtain this vector by setting $x_j = 1$ and the other free variables (for C) equal to 0, and then solving for the remaining (pivot) variables.[14]

On the other hand, x_j is a pivot variable for B; assume that it is the pivot in the ℓ^{th} row. That is, the first nonzero entry of the ℓ^{th} row of B occurs in the j^{th} column. Then the ℓ^{th}

[14] To see why $\mathbf{v}$ has this form, we must understand why the k^{th} entry of $\mathbf{v}$ is 0 whenever $k > j$. So suppose $k > j$. If x_k is a free variable, then by construction the k^{th} entry of $\mathbf{v}$ is 0. On the other hand, if x_k is a pivot variable, then the value of x_k is determined *only* by the values of the pivot variables x_ℓ with $\ell > k$; since, by construction, these are all 0, once again, the k^{th} entry of $\mathbf{v}$ is 0.

entry of $B\mathbf{v}$ is 1. This contradicts Theorem 4.1, for if $C\mathbf{v} = \mathbf{0}$, then $A\mathbf{v} = \mathbf{0}$, and so $B\mathbf{v} = \mathbf{0}$ as well. □

In fact, it is not difficult to see that more is true, as we ask the ambitious reader to check in Exercise 16:

Theorem 4.3. *Each matrix has a unique* reduced *echelon form.*

We conclude with a few examples illustrating Gaussian elimination and its applications.

EXAMPLE 6

Give the general solution of the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + 3x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 + 2x_5 &= -4 \\ x_2 + 2x_3 + 2x_4 - x_5 &= 0 \\ 2x_1 - x_2 + x_4 - 6x_5 &= 9\,. \end{aligned}$$

We begin with the augmented matrix of coefficients and put it in reduced echelon form:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 3 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 2 & -4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 & -6 & 9 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 2 & -4 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -3 & -6 & 3 & -6 & 9 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 3 & -3 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the system of equations is given in reduced echelon form by

$$\begin{aligned} x_1 + x_3 - 2x_5 &= 3 \\ x_2 + 2x_3 + x_5 &= -2 \\ x_4 - x_5 &= 1\,, \end{aligned}$$

from which we read off

$$\begin{aligned} x_1 &= 3 - x_3 + 2x_5 \\ x_2 &= -2 - 2x_3 - x_5 \\ x_3 &= x_3 \\ x_4 &= 1 + x_5 \\ x_5 &= x_5\,, \end{aligned}$$

and so the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

▲

EXAMPLE 7

We wish to find a normal vector to the hyperplane in $\mathbb{R}^4$ spanned by the vectors $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (0, 1, 0, 1)$, and $\mathbf{v}_3 = (1, 2, 3, 4)$. That is, we want a vector $\mathbf{x} \in \mathbb{R}^4$ satisfying the system of equations $\mathbf{v}_1 \cdot \mathbf{x} = \mathbf{v}_2 \cdot \mathbf{x} = \mathbf{v}_3 \cdot \mathbf{x} = 0$. Such a vector $\mathbf{x}$ must satisfy the system of equations

$$\begin{aligned} x_1 \quad\quad + \; x_3 \quad\quad &= 0 \\ x_2 \quad\quad + \; x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0. \end{aligned}$$

Putting the augmented matrix in reduced echelon form, we find

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

From this we read off

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= -x_4 \\ x_3 &= -x_4 \\ x_4 &= x_4, \end{aligned}$$

and so the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix};$$

that is, a normal vector to the plane is (any nonzero scalar multiple of) $(1, -1, -1, 1)$. The reader should check that this vector actually *is* orthogonal to the three given vectors. ▲

Recalling that solving the system of linear equations

$$\mathbf{A}_1 \cdot \mathbf{x} = b_1, \quad \mathbf{A}_2 \cdot \mathbf{x} = b_2, \quad \dots, \quad \mathbf{A}_m \cdot \mathbf{x} = b_m$$

amounts to finding a parametric representation of the intersection of these m hyperplanes, we consider one last example.

EXAMPLE 8

We seek a parametric description of the intersection of the three hyperplanes in $\mathbb{R}^4$ given by

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 - x_3 - 3x_4 &= 7\,. \end{aligned}$$

Again, we start with the augmented matrix and put it in echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -3 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 3 & -3 & -6 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 2 \\ 0 & 3 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right].$$

Without even continuing to reduced echelon form, we see that the new augmented matrix gives the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\ 3x_2 - 3x_3 - 6x_4 &= -3 \\ 0 &= 8\,. \end{aligned}$$

The last equation, $0 = 8$, is, of course, absurd. What happened? There can be no values of x_1, x_2, x_3, and x_4 that make this system of equations hold: The three hyperplanes described by our equations have no point in common. A system of linear equations may not have any solutions; in this case it is called *inconsistent*. We study this notion carefully in the next section. ▲

Exercises 1.4

1. Use elementary operations to find the general solution of each of the following systems of equations. Use the method of Example 1 as a prototype.

a. $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

*b. $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 5x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 0 \end{aligned}$$

c. $$\begin{aligned} 3x_1 - 6x_2 - x_3 + x_4 &= 6 \\ -x_1 + 2x_2 + 2x_3 + 3x_4 &= 3 \\ 4x_1 - 8x_2 - 3x_3 - 2x_4 &= 3 \end{aligned}$$

***2.** Decide which of the following matrices are in echelon form, which are in reduced echelon form, and which are neither. Justify your answers.

a. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

3. For each of the following matrices A, determine its reduced echelon form and give the general solution of $A\mathbf{x} = \mathbf{0}$ in standard form.

*a. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \end{bmatrix}$

*b. $A = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 1 & -2 \\ 3 & -3 & 6 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \\ -1 & 1 & 6 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

*e. $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ -1 & -3 & 1 & 2 & 3 \\ 1 & -1 & 3 & 1 & 1 \\ 2 & -3 & 7 & 3 & 4 \end{bmatrix}$

*g. $A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ -1 & 1 & -1 & 0 & -1 \end{bmatrix}$

h. $A = \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ -1 & 2 & 3 & 4 & 1 & -6 \\ 0 & 4 & 4 & 12 & -1 & -7 \end{bmatrix}$

4. Give the general solution of the equation $A\mathbf{x} = \mathbf{b}$ in standard form.

*a. $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$

*c. $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 4 & 1 & -1 \\ 1 & 2 & 0 & -2 & 2 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 10 \\ -3 \\ 7 \end{bmatrix}$

5. For the following matrices A, give the general solution of the equation $A\mathbf{x} = \mathbf{x}$ in standard form. (*Hint:* Rewrite this as $B\mathbf{x} = \mathbf{0}$ for an appropriate matrix B.)

a. $A = \begin{bmatrix} 10 & -6 \\ 18 & -11 \end{bmatrix}$ *b. $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$ c. $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

6. For the following matrices A, give the general solution of the equation $A\mathbf{x} = 2\mathbf{x}$ in standard form.

a. $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & 16 & -15 \\ 1 & 12 & -9 \\ 1 & 16 & -13 \end{bmatrix}$

7. One might need to find solutions of $A\mathbf{x} = \mathbf{b}$ for several different $\mathbf{b}$'s, say $\mathbf{b}_1, \dots, \mathbf{b}_k$. In this event, one can augment the matrix A with all the $\mathbf{b}$'s simultaneously, forming the "multi-augmented" matrix $[\, A \mid \mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_k \,]$. One can then read off the various solutions from the reduced echelon form of the multi-augmented matrix. Use this method to solve $A\mathbf{x} = \mathbf{b}_j$ for the given matrices A and vectors $\mathbf{b}_j$.

a. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

***8.** Find all the unit vectors $\mathbf{x} \in \mathbb{R}^3$ that make an angle of $\pi/3$ with each of the vectors $(1, 0, -1)$ and $(0, 1, 1)$.

9. Find all the unit vectors $\mathbf{x} \in \mathbb{R}^3$ that make an angle of $\pi/4$ with $(1, 0, 1)$ and an angle of $\pi/3$ with $(0, 1, 0)$.

10. Find a normal vector to the hyperplane in $\mathbb{R}^4$ spanned by

*a. $(1, 1, 1, 1)$, $(1, 2, 1, 2)$, and $(1, 3, 2, 4)$;

b. $(1, 1, 1, 1)$, $(2, 2, 1, 2)$, and $(1, 3, 2, 3)$.

11. Find all vectors $\mathbf{x} \in \mathbb{R}^4$ that are orthogonal to both

*a. $(1, 0, 1, 1)$ and $(0, 1, -1, 2)$;

b. $(1, 1, 1, -1)$ and $(1, 2, -1, 1)$.

12. Find all the unit vectors in $\mathbb{R}^4$ that make an angle of $\pi/3$ with $(1, 1, 1, 1)$ and an angle of $\pi/4$ with both $(1, 1, 0, 0)$ and $(1, 0, 0, 1)$.

♯***13.** Let A be an $m \times n$ matrix, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and let c be a scalar. Show that

a. $A(c\mathbf{x}) = c(A\mathbf{x})$

b. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$

14. Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.

a. Show that if $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ are both solutions of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{u} - \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

b. Suppose $\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Show that $\mathbf{u} + \mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Hint: Use Exercise 13.

15. a. Prove or give a counterexample: If A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$, then either every entry of A is 0 or $\mathbf{x} = \mathbf{0}$.

b. Prove or give a counterexample: If A is an $m \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then every entry of A is 0.

Although an example does not constitute a proof, a *counterexample* is a fine disproof: A counterexample is merely an explicit example illustrating that the statement is false.

Here, the evil authors are asking you first to decide whether the statement is true or false. It is important to try examples to develop your intuition. In a problem like this that contains arbitrary positive integers m and n, it is often good to start with small values. Of course, if we take $m = n = 1$, we get the statement

If a is a real number and $ax = 0$ for every real number x, then $a = 0$.

Here you might say, "Well, if $a \neq 0$, then I can divide both sides of the equation by a and obtain $x = 0$. Since the equation must hold for *all* real numbers x, we must have $a = 0$." But this doesn't give us any insight into the general case, as we can't divide by vectors or matrices.

What are some alternative approaches? You might try picking a particular value of x that will shed light on the situation. For example, if we take $x = 1$, then we immediately get $a = 0$. How might you use this idea to handle the general case? If you wanted to show that a particular entry, say a_{25}, of the matrix A was 0, could you pick the vector $\mathbf{x}$ appropriately?

There's another way to pick a particular value of x that leads to information. Since the only given object in the problem is the real number a, we might try letting $x = a$ and see what happens. Here we get $ax = a^2 = 0$, from which we conclude immediately that $a = 0$. How does this idea help us with the general case? Remember that the entries of the vector $A\mathbf{x}$ are the dot products $\mathbf{A}_i \cdot \mathbf{x}$. Looking back at part *a* of Exercise 1.2.16, we learned there that if $\mathbf{a} \cdot \mathbf{x} = 0$ for all $\mathbf{x}$, then $\mathbf{a} = \mathbf{0}$. How does our current path of reasoning lead us to this?

16. Prove that the reduced echelon form of a matrix is unique, as follows. Suppose B and C are reduced echelon forms of a given nonzero $m \times n$ matrix A.

a. Deduce from the proof of Proposition 4.2 that B and C have the same pivot variables.

b. Explain why the pivots of B and C are in the identical positions. (This is true even without the assumption that the matrices are in *reduced* echelon form.)

c. By considering the solutions in standard form of $B\mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$, deduce that $B = C$.

17. In rating the efficiency of different computer algorithms for solving a system of equations, it is usually considered sufficient to compare the number of multiplications required to carry out the algorithm.

a. Show that

$$n(n-1) + (n-1)(n-2) + \cdots + (2)(1) = \sum_{k=1}^{n}(k^2 - k)$$

multiplications are required to bring a general $n \times n$ matrix to echelon form by (forward) Gaussian elimination.

b. Show that $\sum_{k=1}^{n}(k^2-k) = \frac{1}{3}(n^3-n)$. (*Hint:* For some appropriate formulas, see Exercise 1.2.10.)

c. Now show that it takes $n+(n-1)+(n-2)+\cdots+1 = n(n+1)/2$ multiplications to bring the matrix to reduced echelon form by clearing out the columns above the pivots, working right to left. Show that it therefore takes a total of $\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$ multiplications to put A in reduced echelon form.

d. Gauss-Jordan elimination is a slightly different algorithm used to bring a matrix to reduced echelon form: Here each column is cleared out, both below and above the pivot, before moving on to the next column. Show that in general this procedure requires $n^2(n-1)/2$ multiplications. For large n, which method is preferred?

5 The Theory of Linear Systems

We developed Gaussian elimination as a technique for finding a parametric description of the solutions of a system of linear Cartesian equations. Now we shall see that this same technique allows us to proceed in the opposite direction. That is, given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$, we would like to find a set of Cartesian equations whose solution is precisely $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. In addition, we will rephrase in somewhat more general terms the observations we have already made about solutions of systems of linear equations.

5.1 Existence, Constraint Equations, and Rank

Suppose A is an $m \times n$ matrix. There are two equally important ways to interpret the system of equations $A\mathbf{x} = \mathbf{b}$. In the preceding section, we concentrated on the row vectors of A: If $\mathbf{A}_1, \dots, \mathbf{A}_m$ denote the *row vectors* of A, then the vector $\mathbf{c}$ is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if

$$\mathbf{A}_1 \cdot \mathbf{c} = b_1, \quad \mathbf{A}_2 \cdot \mathbf{c} = b_2, \quad \dots, \quad \mathbf{A}_m \cdot \mathbf{c} = b_m.$$

Geometrically, $\mathbf{c}$ is a solution precisely when it lies in the intersection of all the hyperplanes defined by the system of equations.

On the other hand, we can define the *column vectors* of the $m \times n$ matrix A as follows:

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m, \quad j = 1, 2, \dots, n.$$

We now make an observation that will be crucial in our future work: The matrix product $A\mathbf{x}$ can also be written as

$$(*) \quad A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Thus, a solution $\mathbf{c} = (c_1, \ldots, c_n)$ of the linear system $A\mathbf{x} = \mathbf{b}$ provides scalars $c_1, \ldots, c_n$ so that

$$\mathbf{b} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n.$$

This is our second geometric interpretation of the system of linear equations: A solution $\mathbf{c}$ gives a representation of the vector $\mathbf{b}$ as a linear combination, $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, of the column vectors of A.

EXAMPLE 1

Consider the four vectors

$$\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

Suppose we want to express the vector $\mathbf{b}$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Writing out the expression

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 2 \end{bmatrix},$$

we obtain the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 3 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 2. \end{aligned}$$

In matrix notation, we must solve $A\mathbf{x} = \mathbf{b}$, where the columns of A are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

So we take the augmented matrix to reduced echelon form:

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -2 & -6 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This tells us that the solution is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \quad \text{so} \quad \mathbf{b} = -2\mathbf{v}_1 + 0\mathbf{v}_2 + 3\mathbf{v}_3,$$

which, as the reader can check, works. ▲

Now we modify the preceding example slightly.

EXAMPLE 2

We would like to express the vector

$$\mathbf{b}' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

as a linear combination of the same vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. This then leads analogously to the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

and to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{array}\right],$$

whose echelon form is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The last row of the augmented matrix corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 = 1,$$

which obviously has no solution. Thus, the original system of equations has no solution: The vector $\mathbf{b}'$ in this example cannot be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ▲

These examples lead us to make the following definition.

Definition. If the system of equations $A\mathbf{x} = \mathbf{b}$ has no solutions, the system is said to be *inconsistent*; if it has at least one solution, then it is said to be *consistent*.

A system of equations is consistent precisely when a solution *exists*. We see that the system of equations in Example 2 is inconsistent and the system of equations in Example 1 is consistent. It is easy to recognize an inconsistent system of equations from the echelon form of its augmented matrix: The system is inconsistent precisely when there is an equation that reads

$$0x_1 + 0x_2 + \cdots + 0x_n = c$$

for some nonzero scalar c, i.e., when there is a row in the echelon form of the augmented matrix all of whose entries are 0 except for the rightmost.

Turning this around a bit, let $[\,U \mid \mathbf{c}\,]$ denote an echelon form of the augmented matrix $[\,A \mid \mathbf{b}\,]$. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if any zero row in U corresponds to a zero entry in the vector $\mathbf{c}$.

There are two geometric interpretations of consistency. From the standpoint of row vectors, the system $A\mathbf{x} = \mathbf{b}$ is consistent precisely when the intersection of the hyperplanes

$$\mathbf{A}_1 \cdot \mathbf{x} = b_1, \quad \dots, \quad \mathbf{A}_m \cdot \mathbf{x} = b_m$$

is nonempty. From the point of view of column vectors, the system $A\mathbf{x} = \mathbf{b}$ is consistent precisely when the vector $\mathbf{b}$ can be written as a linear combination of the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of A; in other words, it is consistent when $\mathbf{b} \in \text{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

In the next example, we characterize those vectors $\mathbf{b} \in \mathbb{R}^4$ that can be expressed as a linear combination of the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from Examples 1 and 2.

EXAMPLE 3

For what vectors

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

will the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_2 + x_3 &= b_2 \\ x_1 + x_2 + x_3 &= b_3 \\ 2x_1 + x_2 + 2x_3 &= b_4 \end{aligned}$$

have a solution? We form the augmented matrix $[\,A \mid \mathbf{b}\,]$ and put it in echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & 1 & b_3 \\ 2 & 1 & 2 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & -1 & b_3 - b_1 \\ 0 & -1 & -2 & b_4 - 2b_1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 1 & b_1 - b_3 \\ 0 & 0 & 0 & -b_1 + b_2 - b_3 + b_4 \end{array}\right].$$

We deduce that the original system of equations will have a solution if and only if

$$(**)\qquad -b_1 + b_2 - b_3 + b_4 = 0.$$

That is, the vector $\mathbf{b}$ belongs to Span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ precisely when $\mathbf{b}$ satisfies the *constraint equation* $(**)$. Changing letters slightly, we infer that a Cartesian equation of the hyperplane spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^4$ is $-x_1 + x_2 - x_3 + x_4 = 0$. ▲

EXAMPLE 4

As a further example, we take

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & -3 \\ 3 & -3 & 3 \end{bmatrix},$$

and we look for constraint equations that describe the vectors $\mathbf{b} \in \mathbb{R}^4$ for which $A\mathbf{x} = \mathbf{b}$ is consistent, i.e., all vectors $\mathbf{b}$ that can be expressed as a linear combination of the columns of A.

As before, we consider the augmented matrix $[\, A \mid \mathbf{b} \,]$ and determine an echelon form $[\, U \mid \mathbf{c} \,]$. In order for the system to be consistent, every entry of $\mathbf{c}$ corresponding to a row of 0's in U must be 0 as well:

$$[\, A \mid \mathbf{b} \,] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 3 & 2 & -1 & b_2 \\ 1 & 4 & -3 & b_3 \\ 3 & -3 & 3 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & 5 & -4 & b_2 - 3b_1 \\ 0 & 5 & -4 & b_3 - b_1 \\ 0 & 0 & 0 & b_4 - 3b_1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & 5 & -4 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_1 \end{array}\right].$$

Here we have two rows of 0's in U, so we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ satisfies the two *constraint equations*

$$2b_1 - b_2 + b_3 = 0 \quad \text{and} \quad -3b_1 + b_4 = 0.$$

These equations describe the intersection of two hyperplanes through the origin in $\mathbb{R}^4$ with respective normal vectors $(2, -1, 1, 0)$ and $(-3, 0, 0, 1)$. ▲

Notice that in the last two examples, we have reversed the process of Sections 3 and 4. There we expressed the general solution of a system of linear equations as a linear combination of certain vectors, just as we described lines, planes, and hyperplanes parametrically earlier. Here, starting with the column vectors of the matrix A, we have found the *constraint equations* that a vector $\mathbf{b}$ must satisfy in order to be a linear combination of them (that is, to be in their span). This is the process of determining Cartesian equations of a space that is defined parametrically.

Remark. It is worth noting that since A has different echelon forms, one can arrive at different constraint equations. We will investigate this more deeply in Chapter 3.

EXAMPLE 5

Find a Cartesian equation of the plane in $\mathbb{R}^3$ given parametrically by

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

We ask which vectors $\mathbf{b} = (b_1, b_2, b_3)$ can be written in the form

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This system of equations can be rewritten as

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} b_1 - 1 \\ b_2 - 2 \\ b_3 - 1 \end{bmatrix},$$

and so we want to know when this system of equations is consistent. Reducing the augmented matrix to echelon form, we have

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 - 1 \\ 0 & 1 & b_2 - 2 \\ 1 & 1 & b_3 - 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|l} 1 & 2 & b_1 - 1 \\ 0 & 1 & b_2 - 2 \\ 0 & 0 & b_3 - b_1 + b_2 - 2 \end{array}\right].$$

Thus, the constraint equation is $-b_1 + b_2 + b_3 - 2 = 0$. A Cartesian equation of the given plane is $x_1 - x_2 - x_3 = -2$. ▲

In general, given an $m \times n$ matrix, we might wonder how many conditions a vector $\mathbf{b} \in \mathbb{R}^m$ must satisfy in order to be a linear combination of the columns of A. From the procedure we've just followed, the answer is quite clear: Each row of 0's in the echelon form of A contributes one constraint. This leads us to our next definition.

> ***Definition.*** The *rank* of a matrix A is the number of nonzero rows (the number of pivots) in any echelon form of A. It is usually denoted by r.

Then the number of rows of 0's in the echelon form is $m - r$, and $\mathbf{b}$ must satisfy $m - r$ constraint equations. Note that it is a consequence of Proposition 4.2 that the rank of a matrix is well-defined, i.e., independent of the choice of echelon form.

Now, given a system of m linear equations in n variables, let A denote its coefficient matrix and r the rank of A. We summarize the above remarks as follows.

Proposition 5.1. *The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of the augmented matrix $[\,A \mid \mathbf{b}\,]$ equals the rank of A. In particular, when the rank of A equals m, the system $A\mathbf{x} = \mathbf{b}$ will be consistent for all vectors $\mathbf{b} \in \mathbb{R}^m$.*

Proof. Let $[\,U \mid \mathbf{c}\,]$ denote the echelon form of the augmented matrix $[\,A \mid \mathbf{b}\,]$. We know that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if any zero row in U corresponds to a zero entry in the vector $\mathbf{c}$, which occurs if and only if the number of nonzero rows in the augmented matrix

$[\,U \mid \mathbf{c}\,]$ equals the number of nonzero rows in U, i.e., the rank of A. When $r = m$, there is no row of 0's in U and hence no possibility of inconsistency. □

5.2 Uniqueness and Nonuniqueness of Solutions

We now turn our attention to the question of how many solutions a given *consistent* system of equations has. Our experience with solving systems of equations in Sections 3 and 4 suggests that the solutions of a consistent linear system $A\mathbf{x} = \mathbf{b}$ are intimately related to the solutions of the system $A\mathbf{x} = \mathbf{0}$.

Definition. A system $A\mathbf{x} = \mathbf{b}$ of linear equations is called *inhomogeneous* when $\mathbf{b} \neq \mathbf{0}$; the corresponding equation $A\mathbf{x} = \mathbf{0}$ is called the associated *homogeneous system*.

To relate the solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ and those of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, we need the following fundamental algebraic observation.

Proposition 5.2. *Let A be an $m \times n$ matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then*

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}.$$

(This is called the distributive property *of matrix multiplication.)*

Proof. Recall that, by definition, the i^{th} entry of the product $A\mathbf{x}$ is equal to the dot product $\mathbf{A}_i \cdot \mathbf{x}$. The distributive property of dot product (the last property listed in Proposition 2.1) dictates that

$$\mathbf{A}_i \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A}_i \cdot \mathbf{x} + \mathbf{A}_i \cdot \mathbf{y},$$

and so the i^{th} entry of $\mathbf{A}(\mathbf{x} + \mathbf{y})$ equals the i^{th} entry of $A\mathbf{x} + A\mathbf{y}$. Since this holds for all $i = 1, \dots, m$, the vectors are equal. □

This argument establishes the first part of the following theorem.

Theorem 5.3. *Assume the system $A\mathbf{x} = \mathbf{b}$ is consistent, and let $\mathbf{u}_1$ be a particular solution.*[15] *Then all the solutions are of the form*

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{v}$$

for some solution $\mathbf{v}$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. First we observe that any such vector $\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Using Proposition 5.2, we have

$$A\mathbf{u} = A(\mathbf{u}_1 + \mathbf{v}) = A\mathbf{u}_1 + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Conversely, *every* solution of $A\mathbf{x} = \mathbf{b}$ can be written in this form, for if $\mathbf{u}$ is an arbitrary solution of $A\mathbf{x} = \mathbf{b}$, then, by distributivity again,

$$A(\mathbf{u} - \mathbf{u}_1) = A\mathbf{u} - A\mathbf{u}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{v} = \mathbf{u} - \mathbf{u}_1$ is a solution of the associated homogeneous system; now we just solve for $\mathbf{u}$, obtaining $\mathbf{u} = \mathbf{u}_1 + \mathbf{v}$, as required. □

Remark. As Figure 5.1 suggests, when the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent, its solutions are obtained by *translating* the set of solutions of the associated homogeneous

[15] This is classical terminology for *any* single solution of the inhomogeneous system. There need not be anything special about it. In Example 5 on p. 44, we saw a way to pick a *particular* particular solution.

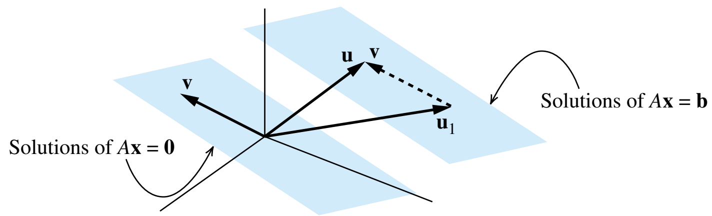


FIGURE 5.1

system by a particular solution $\mathbf{u}_1$. Since $\mathbf{u}_1$ lies on each of the hyperplanes $\mathbf{A}_i \cdot \mathbf{x} = b_i$, $i = 1, \dots, m$, we can translate each of the hyperplanes $\mathbf{A}_i \cdot \mathbf{x} = 0$, which pass through the origin, by the vector $\mathbf{u}_1$. Thus, translating the intersection of the hyperplanes $\mathbf{A}_i \cdot \mathbf{x} = 0$, $i = 1, \dots, m$, by the vector $\mathbf{u}_1$ gives us the intersection of the hyperplanes $\mathbf{A}_i \cdot \mathbf{x} = b_i$, $i = 1, \dots, m$, as indicated in Figure 5.2.

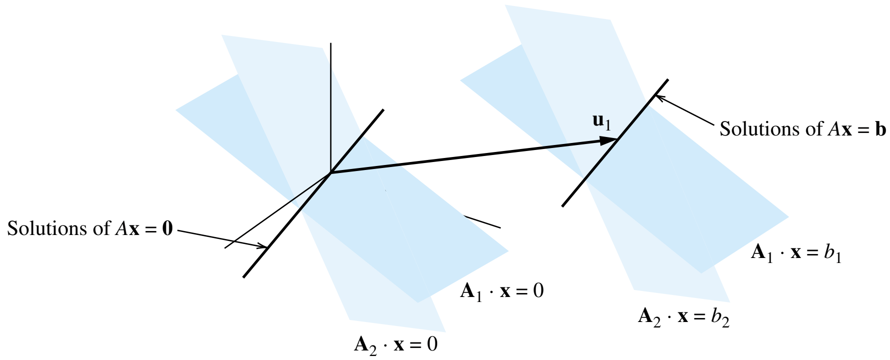


FIGURE 5.2

Of course, a homogeneous system is always consistent, because the *trivial solution*, $\mathbf{x} = \mathbf{0}$, is always a solution of $A\mathbf{x} = \mathbf{0}$. Now, if the rank of A is r, then there will be r pivot variables and $n - r$ free variables in the general solution of $A\mathbf{x} = \mathbf{0}$. In particular, if $r = n$, then $\mathbf{x} = \mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$.

Definition. If the system of equations $A\mathbf{x} = \mathbf{b}$ has precisely one solution, then we say that the system has a *unique* solution.

Thus, a homogeneous system $A\mathbf{x} = \mathbf{0}$ has a *unique* solution when $r = n$ and *infinitely many* solutions when $r < n$. Note that it is impossible to have $r > n$, since there cannot be more pivots than columns. Similarly, there cannot be more pivots than rows in the matrix, so it follows that whenever $n > m$ (i.e., there are more variables than equations), the homogeneous system $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions.

From Theorem 5.3 we know that if the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent, then its solutions are obtained by translating the solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ by a particular solution. So we have the following proposition.

Proposition 5.4. *Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent. Then it has a unique solution if and only if the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens exactly when $r = n$.*

We conclude this discussion with an important special case. It is natural to ask when the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for *every* $\mathbf{b} \in \mathbb{R}^m$. From Proposition 5.1 we infer that for the system always to be consistent, we must have $r = m$; from Proposition 5.4 we infer that for solutions to be unique, we must have $r = n$. And so we see that we can have both conditions only when $r = m = n$.

Definition. An $n \times n$ matrix of rank $r = n$ is called *nonsingular*. An $n \times n$ matrix of rank $r < n$ is called *singular*.

We observe that an $n \times n$ matrix is nonsingular if and only if there is a pivot in each row, hence in each column, of its echelon form. Thus, its reduced echelon form must be the $n \times n$ matrix

$$\begin{bmatrix} 1 & & & & \\ & 1 & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & 1 & \\ & & & & 1 \end{bmatrix}.$$

It seems silly to remark that when $m = n$, if $r = n$, then $r = m$, and conversely. But the following result, which will be extremely important in the next few chapters, is an immediate consequence of this observation.

Proposition 5.5. *Let A be an $n \times n$ matrix. The following are equivalent:*

1. *A is nonsingular.*
2. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
3. *For every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (indeed, a unique solution).*

Exercises 1.5

*1. By solving a system of equations, find the linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ that gives $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

*2. For each of the following vectors $\mathbf{b} \in \mathbb{R}^4$, decide whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

a. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ b. $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ c. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$

3. Find constraint equations (if any) that **b** must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to be consistent.

a. $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix}$

*b. $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

c. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

*d. $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \\ -2 & -1 & 1 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

4. Find constraint equations that **b** must satisfy in order to be an element of

a. $V = \text{Span}\big((-1, 2, 1), (2, -4, -2)\big)$

b. $V = \text{Span}\big((1, 0, 1, 1), (0, 1, 1, 2), (1, 1, 1, 0)\big)$

c. $V = \text{Span}\big((1, 0, 1, 1), (0, 1, 1, 2), (2, -1, 1, 0)\big)$

d. $V = \text{Span}\big((1, 2, 3), (-1, 0, -2), (1, -2, 1)\big)$

5. By finding appropriate constraint equations, give a Cartesian equation of each of the following planes in $\mathbb{R}^3$.

a. $\mathbf{x} = s(1, -2, -2) + t(2, 0, -1),\ s, t \in \mathbb{R}$

b. $\mathbf{x} = (1, 2, 3) + s(1, -2, -2) + t(2, 0, -1),\ s, t \in \mathbb{R}$

c. $\mathbf{x} = (4, 2, 1) + s(1, 0, 1) + t(1, 2, -1),\ s, t \in \mathbb{R}$

6. Suppose A is a 3×4 matrix satisfying the equations

$$A\begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find a vector $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Give your reasoning. (*Hint:* Look carefully at the vectors on the right-hand side of the equations.)

7. Find a matrix A with the given property or explain why none can exist.

a. One of the rows of A is $(1, 0, 1)$, and for some $\mathbf{b} \in \mathbb{R}^2$ both the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}$.

*b. The rows of A are linear combinations of $(0, 1, 0, 1)$ and $(0, 0, 1, 1)$, and for some $\mathbf{b} \in \mathbb{R}^2$ both the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ are solution of the equation $A\mathbf{x} = \mathbf{b}$.

c. The rows of A are orthogonal to $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, and for some nonzero vector $\mathbf{b} \in \mathbb{R}^2$ both the vectors $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}$.

*d. For some vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ both the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\1\end{bmatrix}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}_1$, and both the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}_2$.

***8.** Let $A = \begin{bmatrix}1 & \alpha\\ \alpha & 3\alpha\end{bmatrix}$.

a. For which numbers α will A be singular?

b. For all numbers α *not* on your list in part *a*, we can solve $A\mathbf{x} = \mathbf{b}$ for every vector $\mathbf{b} \in \mathbb{R}^2$. For each of the numbers α on your list, give the vectors $\mathbf{b}$ for which we can solve $A\mathbf{x} = \mathbf{b}$.

9. Let $A = \begin{bmatrix}1 & \alpha & \alpha\\ \alpha & 2 & 1\\ \alpha & \alpha & 1\end{bmatrix}$.

a. For which numbers α will A be singular?

b. For all numbers α *not* on your list in part *a*, we can solve $A\mathbf{x} = \mathbf{b}$ for every vector $\mathbf{b} \in \mathbb{R}^3$. For each of the numbers α on your list, give the vectors $\mathbf{b}$ for which we can solve $A\mathbf{x} = \mathbf{b}$.

10. Let A be an $m \times n$ matrix. Prove or give a counterexample: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{b}$ always has a unique solution.

11. Let A and B be $m \times n$ matrices. Prove or give a counterexample: If $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions, then the set of vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent is the same as the set of the vectors $\mathbf{b}$ such that $B\mathbf{x} = \mathbf{b}$ is consistent.

12. In each case, give positive integers m and n and an example of an $m \times n$ matrix A with the stated property, or explain why none can exist.

*a. $A\mathbf{x} = \mathbf{b}$ is inconsistent for every $\mathbf{b} \in \mathbb{R}^m$.

*b. $A\mathbf{x} = \mathbf{b}$ has one solution for every $\mathbf{b} \in \mathbb{R}^m$.

c. $A\mathbf{x} = \mathbf{b}$ has no solutions for some $\mathbf{b} \in \mathbb{R}^m$ and one solution for every other $\mathbf{b} \in \mathbb{R}^m$.

d. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every $\mathbf{b} \in \mathbb{R}^m$.

*e. $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^m$ and has infinitely many solutions whenever it is consistent.

f. There are vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ so that $A\mathbf{x} = \mathbf{b}_1$ has no solution, $A\mathbf{x} = \mathbf{b}_2$ has exactly one solution, and $A\mathbf{x} = \mathbf{b}_3$ has infinitely many solutions.

♯**13.** Suppose A is an $m \times n$ matrix with rank m and $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are vectors with $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathbb{R}^n$. Prove that $\text{Span}(A\mathbf{v}_1, \dots, A\mathbf{v}_k) = \mathbb{R}^m$.

14. Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{R}^n$.

*a. Suppose $\mathbf{A}_1 + \cdots + \mathbf{A}_m = \mathbf{0}$. Deduce that $\text{rank}(A) < m$. (*Hint:* Why must there be a row of 0's in the echelon form of A?)

b. More generally, suppose there is some linear combination $c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m = \mathbf{0}$, where some $c_i \neq 0$. Show that $\text{rank}(A) < m$.

15. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$.

a. Suppose $\mathbf{a}_1 + \cdots + \mathbf{a}_n = \mathbf{0}$. Prove that $\text{rank}(A) < n$. (*Hint:* Consider solutions of $A\mathbf{x} = \mathbf{0}$.)

b. More generally, suppose there is some linear combination $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$, where some $c_i \neq 0$. Prove that $\text{rank}(A) < n$.

6 Some Applications

We whet the reader's appetite with a few simple applications of systems of linear equations. In later chapters, when we begin to think of matrices as representing functions, we will find further applications of linear algebra.

6.1 Curve Fitting

The first application is to fitting data points to a certain class of curves.

EXAMPLE 1

We want to find the equation of the line passing through the points $(1, 1)$, $(2, 5)$, and $(-2, -11)$. Of course, none of us needs any linear algebra to solve this problem—the point-slope formula will do; but let's proceed anyhow.

We hope to find an equation of the form

$$y = mx + b$$

that is satisfied by each of the three points. (See Figure 6.1.) That gives us a system of

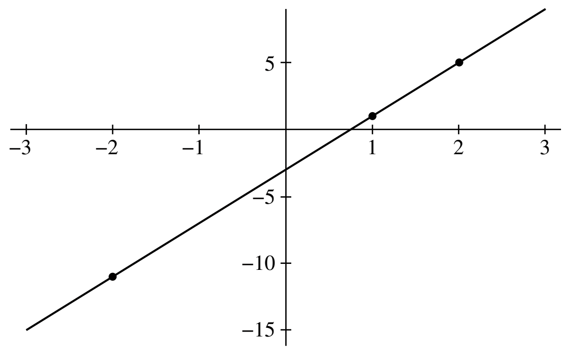


FIGURE 6.1

three equations in the two variables m and b when we substitute the respective points into the equation:

$$\begin{aligned} 1m + b &= 1 \\ 2m + b &= 5 \\ -2m + b &= -11. \end{aligned}$$

It is easy enough to solve this system of equations using Gaussian elimination:

$$\left[\begin{array}{rr|r} 1 & 1 & 1 \\ 2 & 1 & 5 \\ -2 & 1 & -11 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 3 & -9 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right],$$

and so the line we sought is $y = 4x - 3$. The reader should check that all three points indeed lie on this line. ▲

Of course, with three data points, we would expect this system of equations to be inconsistent. In Chapter 4 we will see a beautiful application of dot products and projection to find the line of regression ("least squares line") giving the best fit to the data points in that situation.

Given three points, it is plausible that if they are not collinear, then we should be able to fit a parabola

$$y = ax^2 + bx + c$$

to them (provided no two lie on a vertical line). You are asked to prove this in Exercise 7, but let's do a numerical example here.

EXAMPLE 2

Given the points (0, 3), (2, −5), and (7, 10), we wish to find the parabola $y = ax^2 + bx + c$ passing through them. (See Figure 6.2.) Now we write down the system of equations in

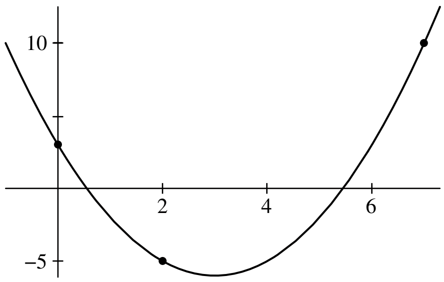


FIGURE 6.2

the variables a, b, and c:

$$\begin{aligned} 0a + 0b + c &= 3 \\ 4a + 2b + c &= -5 \\ 49a + 7b + c &= 10. \end{aligned}$$

We're supposed to solve this system by Gaussian elimination, but we can't resist the temptation to use the fact that $c = 3$ and then rewrite the remaining equations as

$$\begin{aligned} 2a + b &= -4 \\ 7a + b &= 1, \end{aligned}$$

which we can solve easily to obtain $a = 1$ and $b = -6$. Thus, our desired parabola is $y = x^2 - 6x + 3$; once again, the reader should check that each of the three data points lies on this curve. ▲

The curious reader might wonder whether, given $n + 1$ points in the plane (no two with the same x-coordinate), there is a polynomial $P(x)$ of degree at most n so that all $n + 1$ points lie on the graph $y = P(x)$. The answer is yes, as we will prove with the *Lagrange interpolation formula* in Chapter 3. It is widely used in numerical applications.

6.2 Stoichiometry

For our next application, we recall the torture of balancing chemical equations in our freshman chemistry class. The name *stoichiometry*[16] suggests that a chemist should be measuring how many moles of each reactant and product there are; the analysis of how a complicated chemical reaction occurs as a sequence of certain simpler reactions is actually quite fascinating. Nevertheless, there is a fundamental mathematical issue of balancing the number of atoms of each element on the two sides of the reaction, and this amounts to—surprise! surprise!—a system of linear equations.

EXAMPLE 3

Consider the chemical reaction

$$aH_2O + bFe \leftrightharpoons cFe(OH)_3 + dH_2.$$

We wish to find the smallest positive integers a, b, c, and d for which the "equation balances." Comparing the number of atoms of H, O, and Fe on either side of the reaction leads to the following system of equations:

$$\begin{aligned} 2a &= 3c + 2d \\ a &= 3c \\ b &= c \end{aligned}$$

Moving all the variables to the left side yields the homogeneous system

$$\begin{aligned} 2a \quad - 3c - 2d &= 0 \\ a \quad - 3c \quad &= 0 \\ b - c \quad &= 0. \end{aligned}$$

Applying Gaussian elimination to bring the coefficient matrix to reduced echelon form, we find

$$\begin{bmatrix} 2 & 0 & -3 & -2 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \end{bmatrix}.$$

That is, we have

$$\begin{aligned} a &= 2d \\ b &= \tfrac{2}{3}d \\ c &= \tfrac{2}{3}d \\ d &= d, \end{aligned}$$

and so we see that the solution consisting of the smallest positive integers will arise when $d = 3$, resulting in $a = 6$, $b = c = 2$. That is, we have the chemical equation

$$6H_2O + 2Fe \leftrightharpoons 2Fe(OH)_3 + 3H_2.$$ ▲

[16] Indeed, the root is the Greek *stoicheion*, meaning "first principle" or "element."

6.3 Electric Circuits

Electrical appliances, computers, televisions, and so on can be rather simplistically thought of as a network of wires through which electrons "flow," together with various sources of energy providing the impetus to move the electrons. The components of the gadget use the electrons for various purposes (e.g., to provide heat, turn a motor, or light a computer screen) and thus resist the flow of the electrons. The standard unit of measurement of *current* (electron flow) is the amp(ere), that of *resistance* is the ohm, and the unit of *electromotive force* (voltage drop) is the volt. The basic relation among these is given by

(Ohm's Law) $$V = IR,$$

i.e., the electromotive force (often coming from a battery) applied across a wire (in volts) is the product of the current passing through the wire (in amps) and the resistance of the wire (in ohms). For example, a 12-volt battery will create a current of 6 amps in a wire with a 2-ohm resistor.

Now, given a complicated network of wires with resistances and sources of electromotive force, one may ask what current flows in the various wires. Gustav R. Kirchhoff (1824–1887) developed two basic rules with which to answer this question. The first concerns a *node* in the network, a point where two or more wires come together.

> **Kirchhoff's First Law:** The total current coming into a node equals the total current leaving the node.

This is an example of a conservation law. It says that the total number of electrons passing into the node in a given interval of time equals the total number passing out of the node in that same interval of time. It is not hard to see that the first law cannot uniquely determine the currents. For example, given any solution, multiplying all the current values by the same constant will yield another solution.

EXAMPLE 4

Consider the network depicted in Figure 6.3. The three nodes are labeled A, B, and C. The sawlike symbols denote resistors, contributing resistances of $R_1, \dots, R_5$ (in ohms), and the symbol beside the "V" denotes a battery of voltage V. Through the wires flow the currents $I_1, \dots, I_5$ (in amps). Notice that we have indicated a direction of current flow in each wire; by convention, current flows from the "+" side of the battery to the "−" side. If one can't tell in which direction the current will flow, one picks a direction arbitrarily; if the current in a particular wire turns out to be flowing opposite to the direction chosen, then the resulting value of I_j will turn out to be negative. Kirchhoff's first law gives us the three equations

$$\begin{aligned} I_1 - I_2 - I_3 \qquad\qquad &= 0 \\ I_3 - I_4 - I_5 &= 0 \\ -I_1 + I_2 \qquad + I_4 + I_5 &= 0\,. \end{aligned}$$

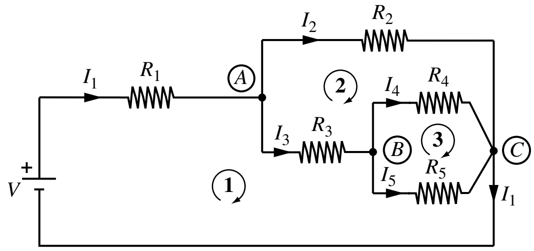


FIGURE 6.3

Since there are fewer equations than unknowns, this system of linear equations must have infinitely many solutions, just as we expect from the physics. Note that the third equation is clearly redundant, so we will discard it below. ▲

To determine the currents completely, we need Kirchhoff's second law, which concerns the *loops* in a network. A loop is any path in the network starting and ending at the same node. We have specified three "basic" loops in this example in the diagram.

Kirchhoff's Second Law: The net voltage drop around a loop is zero. That is, the total "applied" voltage must equal the sum of the products $I_j R_j$ for each wire in the loop.

EXAMPLE 5

We continue the analysis of Example 4. First, loop **1** has an external applied voltage V, so

$$R_1I_1 + R_3I_3 + R_5I_5 = V.$$

(We are writing the variables in this awkward order because we ultimately will be given values of R_j and will want to solve for the currents I_j.) There is no external voltage in loop **2**, so the IR sum must be 0; i.e.,

$$R_2I_2 - R_3I_3 - R_4I_4 = 0.$$

Similarly, loop **3** gives

$$R_4I_4 - R_5I_5 = 0.$$

Of course, there are other loops in the circuit. The reader may find it interesting to notice that any equation determined by another loop is a linear combination of the three equations we've given, so that, in some sense, the three loops we chose are the only ones required.

Summarizing, we now have five equations in the five unknowns $I_1, \dots, I_5$:

$$\begin{array}{rcrcrcrcrcl} I_1 & - & I_2 & - & I_3 & & & & & = & 0 \\ & & & & I_3 & - & I_4 & - & I_5 & = & 0 \\ R_1I_1 & & & + & R_3I_3 & & & + & R_5I_5 & = & V \\ & & R_2I_2 & - & R_3I_3 & + & R_4I_4 & & & = & 0 \\ & & & & & & R_4I_4 & - & R_5I_5 & = & 0. \end{array}$$

One can solve this system for general values of $R_1, \dots, R_5$, and V, but it gets quite messy. Instead, let's assign some specific values and solve the resulting system of linear equations. With

$$R_1 = R_3 = R_4 = 2, \quad R_2 = R_5 = 4, \quad \text{and} \quad V = 210,$$

after some work, we obtain the solution

$$I_1 = 55, \quad I_2 = 25, \quad I_3 = 30, \quad I_4 = 20, \quad I_5 = 10.$$

Here are a few intermediate steps in the row reduction, for those who would like to check our work:

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 2 & 0 & 2 & 0 & 4 & 210 \\ 0 & 4 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 2 & 0 & 2 & 0 & 4 & 210 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 21 & 210 \end{array}\right],$$

from which we deduce that $I_5 = 10$, and the rest follows easily by back-substitution. ▲

As a final remark, we add that this analysis can be applied to other types of network problems where there is a conservation law along with a linear relation between the "force" and the "rate of flow." For example, when water flows in a network of pipes, the amount of water flowing into a joint must equal the amount flowing out (the analogue of Kirchhoff's first law). Also, under a fixed amount of pressure, water will flow faster in a pipe with small cross section than in one with large cross section. Thus, inverse cross-sectional area corresponds to resistance, and water pressure corresponds to voltage; we leave it to the reader to formulate the appropriate version of Kirchhoff's second law. The reader may find it amusing to try to generalize these ideas to trucking networks, airplane scheduling, or manufacturing problems. Also, see Section 5 of Chapter 3, where we take up Kirchhoff's laws from a more conceptual viewpoint.

6.4 Difference Equations and Discrete Dynamical Systems

One of the most extensive applications of linear algebra is to the study of *difference equations*, a discrete version of the differential equations you may have seen in calculus modeling continuous growth and natural phenomena.

EXAMPLE 6

The most basic problem involving population growth comes from simple compounding (so the population is likely to be either rabbits or dollars in a bank account). If x_k denotes the population on day k, we may stipulate that the increase from day k to day $k+1$, i.e.,

$x_{k+1} - x_k$, is given by αx_k, so α represents the daily interest (or reproduction) rate. That is, we have the equation

$$x_{k+1} - x_k = \alpha x_k, \quad \text{or, equivalently,} \quad x_{k+1} = (1+\alpha)x_k,$$

which is easily solved. If x_0 is the original population, then the population on day k is merely

$$x_k = (1+\alpha)^k x_0.$$

No linear algebra there. Now let's consider a problem—admittedly facetious—with two competing species, but more realistic such models play an important role in ecology. Denote by c_k and m_k, respectively, the cat population and the mouse population in month k. Let's say that it has been observed that

$$\begin{aligned} c_{k+1} &= 0.7c_k + 0.2m_k \\ m_{k+1} &= -0.6c_k + 1.4m_k \end{aligned}$$

Note that the presence of mice helps the cat population grow (an ample food supply), whereas the presence of cats diminishes the growth of the mouse population.

Remark. Although this is not in the form of a difference equation per se, any recursive formula of this type can be related to a difference equation simply by subtracting the k^{th} term from both sides. Since the recursion describes how the system changes as time passes, this is called a *discrete dynamical system.*

If we let $\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix}$ denote the cat/mouse population vector in month k, then we have the matrix equation

$$\mathbf{x}_{k+1} = \begin{bmatrix} c_{k+1} \\ m_{k+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix} \begin{bmatrix} c_k \\ m_k \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix} \mathbf{x}_k,$$

and so $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0)$, etc. We can (with the help of a computer) calculate the population some months later. Indeed, it's interesting to see what happens with different beginning cat/mouse populations. (Here we round off to the nearest integer.)

k	c_k	m_k
0	10	25
5	22	49
10	42	89
15	74	152
20	125	254
25	207	418

k	c_k	m_k
0	60	87
5	56	80
10	50	68
15	41	49
20	26	18
25	1	−31

Although the initial populations of $c_0 = 10$ and $m_0 = 25$ allow both species to flourish, there seems to be a catastrophe when the initial populations are $c_0 = 60$ and $m_0 = 87$. The reader should also investigate what happens if we start with, say, 10 cats and 15 mice. We will return to a complete analysis of this example in Chapter 6. ▲

The previous example illustrates how difference equations arise in modeling competition between species. The population of a single species can also be modeled using matrices. For instance, to study the population dynamics of an animal whose life span is 5 years, we can denote by p_1 the number of animals from age 0 to age 1, by p_2 the number of animals from age 1 to age 2, and so on, and set up the vector

$$\mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_5 \end{bmatrix}.$$

From year to year, a certain fraction of each segment of the population will survive to graduate to the next level, and each level will contribute to the first level through some birthrate. Thus, if we let

$$\mathbf{x}_k = \begin{bmatrix} p_1(k) \\ p_2(k) \\ \vdots \\ p_5(k) \end{bmatrix}$$

denote the population distribution after k years, then we will have

$$\begin{aligned}
p_1(k+1) &= b_1 p_1(k) + b_2 p_2(k) + b_3 p_3(k) + b_4 p_4(k) + b_5 p_5(k) \\
p_2(k+1) &= r_1 p_1(k) \\
p_3(k+1) &= r_2 p_2(k) \\
p_4(k+1) &= r_3 p_3(k) \\
p_5(k+1) &= r_4 p_4(k),
\end{aligned}$$

where the coefficients $b_1, \dots, b_5$ are the birthrates for the various population segments and the coefficients $r_1, \dots, r_4$ are the respective graduation rates. We can write the above system of equations in the matrix form $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \end{bmatrix}.$$

The matrix A is called the *Leslie matrix* after P. H. Leslie, who introduced these population distribution studies in the 1940s.

EXAMPLE 7

The flour beetle *Tribolium castaneum* is largely considered a pest, but is a particularly nice species to study for its population distributions. As all beetles do, *Tribolium* goes through three major stages of life: larval, pupal, and adult. The larval and pupal stages are about the same duration (two weeks), and only the adults are reproductive. Thus, if we let $L(k)$, $P(k)$, and $A(k)$ denote the populations of larvae, pupae, and adults after $2k$ weeks, we will have the following system of equations:

$$\begin{bmatrix} L(k+1) \\ P(k+1) \\ A(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & b \\ r_1 & 0 & 0 \\ 0 & r_2 & s \end{bmatrix} \begin{bmatrix} L(k) \\ P(k) \\ A(k) \end{bmatrix},$$

where b denotes the birthrate, r_1 and r_2 denote the graduation rates, respectively, from larvae to pupae and from pupae to adults, and s is the survival rate of the adults from one 2-week period to the next. What makes this species nice for population studies is that by hand culling, it is easy to adjust the rates r_1, r_2, and s (and by introducing additional larvae, one can increase b). For example, if we take $b = 0.9$, $r_1 = 0.9$, $r_2 = 0.8$, and $s = 0.4$ and start with the initial populations $L(0) = P(0) = 0$ and $A(0) = 100$, we get the following results (rounded to the nearest integer):

k	$L(k)$	$P(k)$	$A(k)$
0	0	0	100
10	50	36	61
20	61	53	69
30	75	65	85
40	92	81	105
50	114	100	129

But if we change r_1 to $r_1 = 0.6$, then we get

k	$L(k)$	$P(k)$	$A(k)$
0	0	0	100
10	19	11	21
20	8	5	8
30	3	2	3
40	1	1	1
50	1	0	1

Thus, in this scenario, an effective larvicide can control this pest.[17] ▲

Iterated systems like those that arise in difference equations also arise in other contexts. Here is an example from probability.

EXAMPLE 8

Suppose that over the years in which Fred and Barney have played cribbage, they have observed that when Fred wins a game, he has a 60% chance of winning the next game, whereas when Barney wins a game, he has only a 55% chance of winning the next game. Fred wins the first game; one might wonder what Fred's "expected" win/loss ratio will be after 5 games, 10 games, 100 games, and so on. And how would this have changed if Barney had won the first game?

It is somewhat surprising that this too is a problem in linear algebra. The reason is that there is a system of two linear equations lurking here: If p_k is the probability that Fred wins the k^{th} game and $q_k = 1 - p_k$ is the probability that Barney wins the k^{th} game, then what

[17] More elaborate models have been developed for *Tribolium* involving nonlinear effects, such as the cannibalistic tendency of adults to eat pupae and eggs. These models show that the population can display fascinating dynamics such as periodicity and even chaos.

can we say about p_{k+1} and q_{k+1}? The basic formula from probability theory is this:

$$p_{k+1} = 0.60p_k + 0.45q_k$$
$$q_{k+1} = 0.40p_k + 0.55q_k$$

The probability that Fred wins the game is 0.60 if he won the preceding game and only 0.45 if he lost. Reciprocally, the probability that Barney wins the game is 0.40 if he lost the preceding game (i.e., Fred won) and 0.55 if he won. We can write this system of linear equations in matrix and vector notation if we let

$$\mathbf{x}_k = \begin{bmatrix} p_k \\ q_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0.60 & 0.45 \\ 0.40 & 0.55 \end{bmatrix},$$

for then we have

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.60 & 0.45 \\ 0.40 & 0.55 \end{bmatrix} \mathbf{x}_k, \quad k = 1, 2, 3, \ldots.$$

If Fred wins the first game, we have $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and we can calculate (to five decimal places only)

$$\begin{aligned}
\mathbf{x}_2 = A\mathbf{x}_1 &= \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} \\
\mathbf{x}_3 = A\mathbf{x}_2 &= \begin{bmatrix} 0.540 \\ 0.460 \end{bmatrix} \\
\mathbf{x}_4 = A\mathbf{x}_3 &= \begin{bmatrix} 0.531 \\ 0.469 \end{bmatrix} \\
\mathbf{x}_5 = A\mathbf{x}_4 &= \begin{bmatrix} 0.52965 \\ 0.47035 \end{bmatrix} \\
&\vdots \\
\mathbf{x}_{10} = A\mathbf{x}_9 &= \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix} \\
&\vdots \\
\mathbf{x}_{100} = A\mathbf{x}_{99} &= \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix}.
\end{aligned}$$

These numbers suggest that, provided he wins the first game, Fred has a long-term 52.94% chance of winning any given match in the future.

But what if he loses the first game? Then we take $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and repeat the calculations, arriving at

$$\begin{aligned}
\mathbf{x}_2 = A\mathbf{x}_1 &= \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} \\
\mathbf{x}_3 = A\mathbf{x}_2 &= \begin{bmatrix} 0.5175 \\ 0.4825 \end{bmatrix} \\
\mathbf{x}_4 = A\mathbf{x}_3 &= \begin{bmatrix} 0.52763 \\ 0.47238 \end{bmatrix} \\
\mathbf{x}_5 = A\mathbf{x}_4 &= \begin{bmatrix} 0.52914 \\ 0.47086 \end{bmatrix} \\
&\vdots \\
\mathbf{x}_{10} = A\mathbf{x}_9 &= \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix} \\
&\vdots \\
\mathbf{x}_{100} = A\mathbf{x}_{99} &= \begin{bmatrix} 0.52941 \\ 0.47059 \end{bmatrix}.
\end{aligned}$$

Hmm ... The results of the first match seem irrelevant in the long run. In both cases, it seems clear that the vectors $\mathbf{x}_k$ are approaching a limiting vector $\mathbf{x}_\infty$. Since $A\mathbf{x}_k = \mathbf{x}_{k+1}$, it follows that $A\mathbf{x}_\infty = \mathbf{x}_\infty$. Such a vector $\mathbf{x}_\infty$ is called an *eigenvector* of the matrix A. We'll deal with better ways of computing and understanding this example in Chapter 6. ▲

Perhaps even more surprising, here is an application of these ideas to number theory.

EXAMPLE 9

Consider the famous Fibonacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \dots,$$

where each term (starting with the third) is obtained by adding the preceding two. We will see in Chapter 6 how to use linear algebra to give a concrete formula for the k^{th} number in this sequence. But let's suggest now how this might be the case. If we make vectors out of consecutive pairs of Fibonacci numbers, we get the following:

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$
$$\mathbf{x}_4 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \quad \dots, \quad \mathbf{x}_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix},$$

where a_k is the k^{th} number in the sequence. But the rule

$$a_{k+2} = a_k + a_{k+1}$$

allows us to give a "transition matrix" that turns each vector in this list into the next. For example,

$$\mathbf{x}_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x}_2;$$

$$\mathbf{x}_4 = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 3+5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x}_3,$$

and so on. Since

$$\mathbf{x}_{k+1} = \begin{bmatrix} a_{k+1} \\ a_{k+2} \end{bmatrix} = \begin{bmatrix} a_{k+1} \\ a_k + a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x}_k,$$

we see that, as in the previous examples, by repeatedly multiplying our original vector by the transition matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, we can get as far as we'd like in the sequence. ▲

Exercises 1.6

*1. (from Henry Burchard Fine's *A College Algebra*, 1905) A and B are alloys of silver and copper. An alloy that is 5 parts A and 3 parts B is 52% silver. One that is 5 parts A and 11 parts B is 42% silver. What are the percentages of silver in A and B, respectively?

2. (from Henry Burchard Fine's *A College Algebra*, 1905) Two vessels, A and B, contain mixtures of alcohol and water. A mixture of 3 parts from A and 2 parts from B will contain 40% of alcohol; and a mixture of 1 part from A and 2 parts from B will contain 32% of alcohol. What are the percentages of alcohol in A and B, respectively?

3. (from Henry Burchard Fine's *A College Algebra*, 1905) Two points move at constant rates along the circumference of a circle whose length is 150 ft. When they move in opposite senses they meet every 5 seconds; when they move in the same sense they are together every 25 seconds. What are their rates?

4. A grocer mixes dark roast and light roast coffee beans to sell what she calls a French blend and a Viennese blend. For French blend she uses a mixture that is 3 parts dark and 1 part light roast; for Viennese, she uses a mixture that is 1 part dark and 1 part light roast. If she has at hand 20 pounds of dark roast and 17 pounds of light roast, how many pounds each of French and Viennese blend can she make so as to have no waste?

*5. Find the parabola $y = ax^2 + bx + c$ passing through the points $(-1, 9)$, $(1, -1)$, and $(2, 3)$.

6. Find the parabola $y = ax^2 + bx + c$ passing through the points $(-2, -6)$, $(1, 6)$, and $(3, 4)$.

7. Let $P_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2, 3$. Assume x_1, x_2, and x_3 are distinct (i.e., no two are equal).
   a. Show that the matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

   is nonsingular.[18]

[18] No confusion intended here: x_i^2 means $(x_i)^2$, i.e., the square of the real number x_i.

b. Show that the system of equations

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

always has a unique solution. (*Hint:* Try reordering a, b, and c.)

Remark. If $a = 0$, then the points P_1, P_2, and P_3 lie on the line $y = bx + c$; thus, we have shown that three noncollinear points lie on a unique parabola $y = ax^2 + bx + c$.

8. Find the cubic $y = ax^3 + bx^2 + cx + d$ passing through the points $(-2, 5)$, $(-1, -3)$, $(1, -1)$, and $(2, -3)$.

***9.** A circle C passes through the points $(2, 6)$, $(-1, 7)$, and $(-4, -2)$. Find the center and radius of C. (*Hint:* The equation of a circle can be written in the form $x^2 + y^2 + ax + by + c = 0$. Why?)

10. A circle C passes through the points $(-7, -2)$, $(-1, 4)$, and $(1, 2)$. Find the center and radius of C.

11. Let $P_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2, 3$. Let

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

a. Show that the three points P_1, P_2, and P_3 are collinear if and only if the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. (*Hint:* A general line in $\mathbb{R}^2$ is of the form $ax + by + c = 0$, where a and b are not both 0.)

b. Deduce that if the three given points are not collinear, then there is a unique circle passing through them. (*Hint:* If you set up a system of linear equations as suggested by the hint for Exercise 9, you should use part *a* to deduce that the appropriate coefficient matrix is nonsingular.)

12. Use Gaussian elimination to balance the following chemical reactions.

*a. $a\mathrm{Cl_2} + b\mathrm{KOH} \leftrightharpoons c\mathrm{KCl} + d\mathrm{KClO_3} + e\mathrm{H_2O}$

b. $a\mathrm{Pb(N_3)_2} + b\mathrm{Cr(MnO_4)_2} \leftrightharpoons c\mathrm{Cr_2O_3} + d\mathrm{MnO_2} + e\mathrm{Pb_3O_4} + f\mathrm{N_2}$

13. Use Gaussian elimination to solve for the following partial fraction decompositions.

*a. $\dfrac{4x^3 - 7x}{x^4 - 5x^2 + 4} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x-2} + \dfrac{D}{x+2}$

b. $\dfrac{x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$

***14.** In each of the circuits pictured in Figure 6.4, calculate the current in each of the wires. (Ω is the standard abbreviation for ohms.)

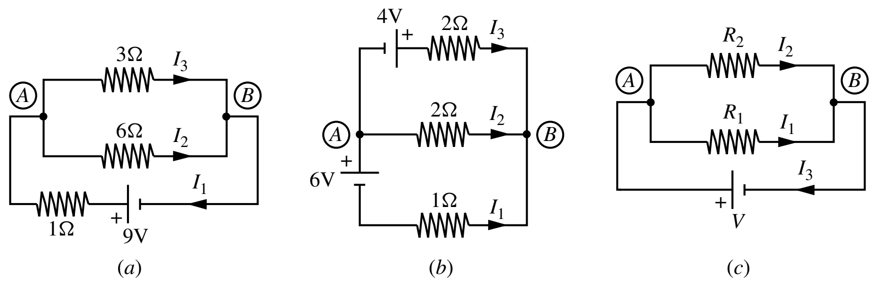


FIGURE 6.4

15. Let A be the matrix given in Example 8.

a. Show that we can find vectors $\mathbf{x} \in \mathbb{R}^2$ satisfying $A\mathbf{x} = \mathbf{x}$ by solving $B\mathbf{x} = \mathbf{0}$, where

$$B = \begin{bmatrix} -0.40 & 0.45 \\ 0.40 & -0.45 \end{bmatrix}.$$

(See Exercise 1.4.5.) Give the general solution of $B\mathbf{x} = \mathbf{0}$ in standard form.

b. Find the solution $\mathbf{x}$ of $B\mathbf{x} = \mathbf{0}$ with $x_1 + x_2 = 1$. (Note that in our discussion of Example 8, we always had $p_k + q_k = 1$.)

c. Compare your answer to part *b* with the vector $\mathbf{x}_\infty$ obtained in Example 8.

16. Investigate (with a computer or programmable calculator) the cat/mouse population behavior in Example 6, choosing a variety of beginning populations, if

a. $$\begin{aligned} c_{k+1} &= 0.7c_k + 0.1m_k \\ m_{k+1} &= -0.2c_k + m_k \end{aligned}$$

*b. $$\begin{aligned} c_{k+1} &= 1.3c_k + 0.2m_k \\ m_{k+1} &= -0.1c_k + m_k \end{aligned}$$

c. $$\begin{aligned} c_{k+1} &= 1.1c_k + 0.3m_k \\ m_{k+1} &= 0.1c_k + 0.9m_k \end{aligned}$$

17. Suppose a living organism that can live to a maximum age of 3 years has Leslie matrix

$$A = \begin{bmatrix} 0 & 0 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}.$$

Find a stable age distribution vector $\mathbf{x}$, i.e., a vector $\mathbf{x} \in \mathbb{R}^3$ with $A\mathbf{x} = \mathbf{x}$.

18. In Example 9, we took our initial vector to be $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a. Find the first ten terms of the sequence obtained by starting instead with $\mathbf{x}'_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

b. Describe the sequence obtained by starting instead with $\mathbf{x}'_0 = \begin{bmatrix} b \\ c \end{bmatrix}$. (*Hint:* Use the fact that $\begin{bmatrix} b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.)

19. (A computer or calculator may be helpful in solving this problem.) Find numbers a, b, c, d, e, and f so that the five points $(0, 2)$, $(-3, 0)$, $(1, 5)$, $(1, 1)$, and $(-1, 1)$ all lie on the conic

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Show, moreover, that a, b, c, d, e, and f are uniquely determined up to a common factor.

HISTORICAL NOTES

In writing this text, we have tried to present the material in a logical manner that builds on ideas in a fairly *linear* fashion. Of course, the historical development of the subject did not go so smoothly. As with any area of mathematics, the ideas and concepts we present in this text were conceived in fits and starts by many different people, at different times, and for entirely different reasons. Only with hindsight were people able to notice patterns and common threads among earlier developments, and gradually a deeper understanding developed. Some computational elements of linear algebra can be traced to civilizations that existed several millennia ago. Throughout history, systems of linear equations have arisen repeatedly in applications. However, the approach we take in today's linear algebra course began to develop in the seventeenth century and did not achieve its polished state until the middle of the twentieth. In these historical notes at the end of each chapter, we will mention some of the mathematicians and scientists who played key roles in the development of linear algebra, and we will outline a few of the routes taken in that development.

The two central topics of this first chapter are vectors and systems of linear equations. The idea of vector, that of a quantity possessing both magnitude and direction, arose in the study of mechanics and forces. Sir Isaac Newton (1642–1727) is credited with the formulation of our current view of forces in his work *Principia* (1687). Pierre de Fermat (1601–1665) and René Descartes (1596–1650) had already laid the groundwork for analytic geometry. Although Fermat published very little of the mathematics he developed, he is generally given credit for having simultaneously developed the ideas that Descartes published in *La Géométrie* (1637).

Following Newton, many scholars began to use directed line segments to represent forces and the parallelogram rule to add such segments. Joseph-Louis Lagrange (1736–1813) published his *Mécanique analytique* in 1788, in which he summarized all the post-Newtonian efforts in a single cohesive mathematical treatise on forces and mechanics. Later, another French mathematician, Louis Poinsot (1777–1859), took the geometry of vector forces to yet another level in his *Éléments de statique* and his subsequent work, in which he invented the geometric study of statics.

Some of these ideas, however, had first appeared 2000 years before Newton. In *Physics*, Aristotle (384–322 BCE) essentially introduced the notion of vector by discussing a force in terms of the distance and direction it displaced an object. Aristotle's work is purely descriptive and not mathematical. In a later work, *Mechanics*, often credited to Aristotle but now believed to be the work of scholarly peers, we find the following insightful observation:

> *Now whenever a body is moved in two directions in a fixed ratio, it necessarily travels in a straight line, which is the diagonal of the figure which the lines arranged in this ratio describe.*

This is, of course, the parallelogram rule.

As for systems of linear equations, virtually all historians cite a particular excerpt from the *Nine Chapters of the Mathematical Art* as the earliest use of what would come to be the modern method for solving such systems, elimination. This text, written during the years 200–100 BCE, at the beginning of the intellectually fruitful Han dynasty, represents the state of Chinese mathematics at that time.

> *There are three types of corn. One bundle of the first type, two of the second, and three of the third total to* 26 *measures. Two bundles of the first type, three of the second, and one of the third total* 34 *measures. Lastly, three bundles of corn of the first type, two bundles of the second type, and one bundle of the third type make a total of* 39 *measures. How many measures make up a single bundle of each type?*

Yes, even back then they had the dreaded word problem!

Today, we would solve this problem by letting x, y, and z represent the measures in a single bundle of each type of corn, translating the word problem to the system of linear equations

$$\begin{aligned} x + 2y + 3z &= 26 \\ 2x + 3y + z &= 34 \\ 3x + 2y + z &= 39. \end{aligned}$$

The Chinese solved the problem in the same way, although they did not use variables, but what is remarkable about the solution given in the *Nine Chapters* is how forward-thinking it truly is. The author wrote the coefficients above as columns in an array

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39. \end{matrix}$$

He then multiplied the first column by 3 and subtracted the third column to put a 0 in the first column (that is, to eliminate x). Similar computations were applied to the second column, and so on. Although the words and formalisms were not there, the ancient Chinese had, in fact, invented matrices and the methods of elimination.

Carl Friedrich Gauss (1777–1855) devised the formal algorithm now known as Gaussian elimination in the early nineteenth century while studying the orbits of asteroids. Wilhelm Jordan (1842–1899) extended Gauss's technique to what is now called Gauss-Jordan elimination in the third edition of his *Handbuch der Vermessungskunde* (1888). Celestial mechanics also led Carl Gustav Jacob Jacobi (1804–1851), a contemporary of Gauss, to a different method of solving the system $A\mathbf{x} = \mathbf{b}$ for certain square matrices A. Jacobi's method gives a way to approximate the solution when A satisfies certain conditions. Other iterative methods, generalizing Jacobi's, have proven invaluable for solving large systems on computers.

Indeed, the modern history of systems of equations has been greatly affected by the advent of the computer age. Problems that were computationally impossible 50 years ago became tractable in the 1960s and 1970s on large mainframes and are now quite manageable on laptops. In this realm, important research continues to find new efficient and effective numerical schemes for solving systems of equations.

CHAPTER 2

MATRIX ALGEBRA

In the previous chapter we introduced matrices as a shorthand device for representing systems of linear equations. Now we will see that matrices have a life of their own, first algebraically and then geometrically. The crucial new ingredient is to interpret an $m \times n$ matrix as a special sort of function that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ the product $A\mathbf{x} \in \mathbb{R}^m$.

1 Matrix Operations

Recall that an $m \times n$ matrix A is a rectangular array of mn real numbers,

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij} represents the *entry* in the i^{th} row and j^{th} column. We recall that two $m \times n$ matrices A and B are equal if $a_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

> We take this opportunity to warn our readers that the word *if* is ordinarily used in mathematical definitions, even though it should be the phrase *if and only if*. That is, even though we don't say so, we intend it to be understood that, for example, in this case, if $A = B$, then $a_{ij} = b_{ij}$ for all i and j. Be warned: This custom applies only to definitions, not to propositions and theorems! See the earlier discussions of *if and only if* on p. 21.

A has m row vectors,

$$\begin{aligned} \mathbf{A}_1 &= (a_{11}, \dots, a_{1n}), \\ \mathbf{A}_2 &= (a_{21}, \dots, a_{2n}), \\ &\vdots \\ \mathbf{A}_m &= (a_{m1}, \dots, a_{mn}), \end{aligned}$$

which are vectors in $\mathbb{R}^n$, and n column vectors,

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

which are, correspondingly, vectors in $\mathbb{R}^m$.

We denote by O the *zero matrix*, the $m \times n$ matrix all of whose entries are 0. We also introduce the notation $\mathcal{M}_{m\times n}$ for the set of all $m \times n$ matrices. For future reference, we call a matrix *square* if $m = n$ (i.e., it has equal numbers of rows and columns). In the case of a square matrix, we refer to the entries a_{ii}, $i = 1, \dots, n$, as *diagonal* entries.

Definition. Let A be an $n \times n$ (square) matrix with entries a_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, n$.

1. We call A *diagonal* if every nondiagonal entry is zero, i.e., if $a_{ij} = 0$ whenever $i \neq j$.
2. We call A *upper triangular* if all of the entries below the diagonal are zero, i.e., if $a_{ij} = 0$ whenever $i > j$.
3. We call A *lower triangular* if all of the entries above the diagonal are zero, i.e., if $a_{ij} = 0$ whenever $i < j$.

Let's now consider various algebraic operations we can perform on matrices. Given an $m \times n$ matrix A, the simplest algebraic manipulation is to multiply every entry of A by a real number c (*scalar multiplication*). If A is the matrix with entries a_{ij} ($i = 1, \dots, m$ and $j = 1, \dots, n$), then cA is the matrix whose entries are ca_{ij}:

$$cA = c\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ ca_{21} & \dots & ca_{2n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix}.$$

Next comes *addition of matrices*. Given $m \times n$ matrices A and B, we define their sum entry by entry. In symbols, when

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ b_{21} & \dots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix},$$

we define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \dots & a_{2n} + b_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}.$$

It is important to understand that when we refer to the set of all $m \times n$ matrices, $\mathcal{M}_{m\times n}$, we have not specified the positive integers m and n. They can be chosen arbitrarily. However, when we say that $A, B \in \mathcal{M}_{m\times n}$, we mean that A and B must have the same "shape," i.e., the same number of rows (m) and the same number of columns (n).

EXAMPLE 1

Let $c = -2$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 4 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 4 & -1 \\ -3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Then

$$cA = \begin{bmatrix} -2 & -4 & -6 \\ -4 & -2 & 4 \\ -8 & 2 & -6 \end{bmatrix}, \quad A + B = \begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & -1 \\ 4 & -1 & 3 \end{bmatrix},$$

and neither sum $A + C$ nor $B + C$ makes sense, because C has a different shape from A and B. ▲

We leave it to the reader to check that scalar multiplication of matrices and matrix addition satisfy the same list of properties we gave in Exercise 1.1.28 for scalar multiplication of vectors and vector addition. We list them here for reference.

Proposition 1.1. *Let $A, B, C \in \mathcal{M}_{m\times n}$ and let $c, d \in \mathbb{R}$.*

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$.
3. $\mathrm{O} + A = A$.
4. *There is a matrix $-A$ so that $A + (-A) = \mathrm{O}$.*
5. $c(dA) = (cd)A$.
6. $c(A + B) = cA + cB$.
7. $(c + d)A = cA + dA$.
8. $1A = A$.

Proof. Left to the reader in Exercise 3. □

To understand these properties, one might simply examine corresponding entries of the appropriate matrices and use the relevant properties of real numbers to see why they are equal. A more elegant approach is the following: We can encode an $m \times n$ matrix as a vector in $\mathbb{R}^{mn}$, for example,

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -5 & 4 \end{bmatrix} \in \mathcal{M}_{3\times 2} \leftrightsquigarrow (1, -1, 2, 3, -5, 4) \in \mathbb{R}^6,$$

and you can check that scalar multiplication and addition of matrices correspond exactly to scalar multiplication and addition of vectors. We will make this concept more precise in Section 6 of Chapter 3.

The real power of matrices comes from the operation of matrix multiplication. Just as we can compute a dot product of two vectors in $\mathbb{R}^n$, ending up with a scalar, we shall see that we can multiply matrices of appropriate shapes:

$$\mathcal{M}_{m\times n} \times \mathcal{M}_{n\times p} \to \mathcal{M}_{m\times p}.$$

In particular, when $m = n = p$ (so that our matrices are square and of the same size), we have a way of combining two $n \times n$ matrices to obtain another $n \times n$ matrix.

Definition. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Their product AB is an $m \times p$ matrix whose ij-entry is

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}\,;$$

that is, the dot product of the i^{th} row vector of A and the j^{th} column vector of B, both of which are vectors in $\mathbb{R}^n$. Graphically, we have

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & & b_{1j} & & b_{1p} \\ b_{21} & & b_{2j} & & b_{2p} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ b_{n1} & & b_{nj} & & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \vdots & & \\ \cdots & \cdots & (AB)_{ij} & \cdots & \cdots \\ & & \vdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

We reiterate that in order for the product AB to be defined, the number of *columns* of A must equal the number of *rows* of B.

Recall that in Section 4 of Chapter 1 we defined the product of an $m \times n$ matrix A with a vector $\mathbf{x} \in \mathbb{R}^n$. The definition we just gave generalizes that if we think of an $n \times p$ matrix B as a collection of p column vectors. In particular,

The j^{th} column of AB is the product of A with the j^{th} column vector of B.

EXAMPLE 2

Note that this definition is compatible with our definition in Chapter 1 of the multiplication of an $m \times n$ matrix with a column vector in $\mathbb{R}^n$ (an $n \times 1$ matrix). For example, if

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 4 & 1 & 0 & -2 \\ -1 & 1 & 5 & 1 \end{bmatrix},$$

then

$$A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}, \quad \text{and}$$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 10 & -2 \\ -1 & 15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 15 & 1 \\ 9 & 1 & -5 & -5 \\ 3 & 2 & 5 & -1 \end{bmatrix}.$$

Notice also that the product BA does not make sense: B is a 2×4 matrix and A is 3×2, and $4 \neq 3$. ▲

The preceding example brings out an important point about the nature of matrix multiplication: It can happen that the matrix product AB is defined and the product BA is not. Now if A is an $m \times n$ matrix and B is an $n \times m$ matrix, then both products AB and BA make sense: AB is $m \times m$ and BA is $n \times n$. Notice that these are both square matrices, but of different sizes.

EXAMPLE 3

To see an extreme example of this, consider the 1×3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and the 3×1 matrix $B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}, \quad \text{whereas}$$

$$BA = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}.$$

▲

Even if we start with both A and B as $n \times n$ matrices, the products AB and BA have the same shape but need not be equal.

EXAMPLE 4

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}, \quad \text{whereas} \quad BA = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}.$$

▲

When—and only when—A is a square matrix, we can multiply A by itself, obtaining $A^2 = AA$, $A^3 = A^2A = AA^2$, etc. In the last examples of Chapter 1, Section 6, the vectors $\mathbf{x}_k$ are obtained from the initial vector $\mathbf{x}_0$ by repeatedly multiplying by the matrix A, so that $\mathbf{x}_k = A^k\mathbf{x}_0$.

EXAMPLE 5

There is an interesting way to interpret matrix powers in terms of directed graphs. Starting with the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

we draw a graph with 3 nodes (vertices) and a_{ij} directed edges (paths) from node i to node j, as shown in Figure 1.1. For example, there are 2 edges from node 1 to node 2 and none from node 3 to node 2. If we multiply a_{ij} by a_{jk}, we get the number of two-step paths from node i to node k passing through node j. Thus, in this case, the sum

$$a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k}$$

gives all the two-step paths from node i to node k. For example, the 13-entry of A^2,

$$(A^2)_{13} = a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = (0)(1) + (2)(1) + (1)(1) = 3,$$

gives the number of two-step paths from node 1 to node 3. With a bit of thought, the reader will convince herself that the ij-entry of A^n is the number of n-step directed paths from node i to node j.

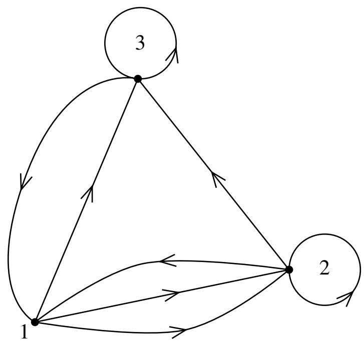


FIGURE 1.1

We calculate

$$A^2 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 3 & 3 \\ 1 & 2 & 2 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 5 & 8 & 8 \\ 6 & 7 & 8 \\ 4 & 4 & 5 \end{bmatrix}, \quad \text{and} \quad \dots$$

$$A^7 = \begin{bmatrix} 272 & 338 & 377 \\ 273 & 337 & 377 \\ 169 & 208 & 233 \end{bmatrix}.$$

In particular, there are 169 seven-step paths from node 3 to node 1. ▲

We have seen that, in general, matrix multiplication is *not* commutative. However, it does have the following crucial properties. Let I_n denote the $n \times n$ matrix with 1's on the diagonal and 0's elsewhere, as illustrated on p. 61.

Proposition 1.2. *Let A and A' be $m \times n$ matrices, let B and B' be $n \times p$ matrices, let C be a $p \times q$ matrix, and let c be a scalar. Then*

1. $AI_n = A = I_m A$. *For this reason, I_n is called the $n \times n$* identity matrix.
2. $(A + A')B = AB + A'B$ *and* $A(B + B') = AB + AB'$. *This is the* distributive *property of matrix multiplication over matrix addition.*
3. $(cA)B = c(AB) = A(cB)$.
4. $(AB)C = A(BC)$. *This is the* associative *property of matrix multiplication.*

Proof. We prove the associative property and leave the rest to the reader in Exercise 4. Note first of all that there is hope: AB is an $m \times p$ matrix and C is a $p \times q$ matrix, so $(AB)C$ will be an $m \times q$ matrix; similarly, A is an $m \times n$ matrix and BC is a $n \times q$ matrix, so $A(BC)$ will be an $m \times q$ matrix. Associativity amounts to the statement that

$$(AB)\mathbf{c} = A(B\mathbf{c})$$

for any column vector $\mathbf{c}$ of the matrix C: To calculate the j^{th} column of $(AB)C$ we multiply AB by the j^{th} column of C; to calculate the j^{th} column of $A(BC)$ we multiply A by the j^{th} column of BC, which, in turn, is the product of B with the j^{th} column of C.

Letting $\mathbf{b}_1, \dots, \mathbf{b}_p$ denote the column vectors of B, we recall (see the crucial observation (∗) on p. 53) that $B\mathbf{c}$ is the linear combination $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$, and so (using Proposition 5.2 of Chapter 1)

$$\begin{aligned} A(B\mathbf{c}) &= A(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p) = c_1(A\mathbf{b}_1) + c_2(A\mathbf{b}_2) + \cdots + c_p(A\mathbf{b}_p) \\ &= c_1(\text{first column of } AB) + c_2(\text{second column of } AB) \\ &\quad + \cdots + c_p(p^{\text{th}} \text{ column of } AB) \\ &= (AB)\mathbf{c}. \end{aligned}$$ □

There is an important conceptual point underlying this computation, as we now study. Through Chapter 1, we thought of matrices simply as an algebraic shorthand for dealing with systems of linear equations. However, we can interpret matrices as functions, hence imparting to them a geometric interpretation and explaining the meaning of matrix multiplication.

Multiplying the $m \times n$ matrix A by vectors $\mathbf{x} \in \mathbb{R}^n$ defines a function

$$\mu_A : \mathbb{R}^n \to \mathbb{R}^m, \quad \text{given by} \quad \mu_A(\mathbf{x}) = A\mathbf{x}.$$

The function μ_A has *domain* $\mathbb{R}^n$ and *range* $\mathbb{R}^m$, and we often say that "μ_A maps $\mathbb{R}^n$ to $\mathbb{R}^m$."

A function $f: X \to Y$ is a "rule" that assigns to each element x of the domain X an element $f(x)$ of the range Y. We refer to $f(x)$ as the *value* of the function at x. We can think of a function as a machine that turns raw ingredients (inputs) into products (outputs), depicted by a diagram such as on the left in Figure 1.2. In high school mathematics and calculus classes, we tend to visualize a function f by means of its graph, the set of ordered pairs (x, y) with $y = f(x)$. The graph must pass the "vertical line test": For each $x = x_0$ in X, there must be exactly one point (x_0, y) among the ordered pairs.

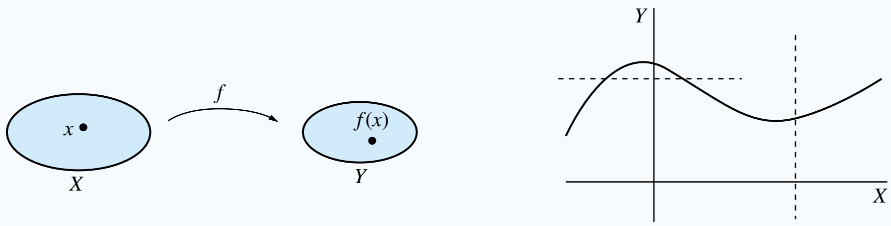


FIGURE 1.2

We say the function is one-to-one if the graph passes the "horizontal line test": For each $y = y_0 \in Y$, there is *at most* one point (x, y_0) among the ordered pairs. The function whose graph is pictured on the right in Figure 1.2 is not one-to-one. More formally, $f: X \to Y$ is *one-to-one* (or injective) if, for $a, b \in X$, the only way we can have $f(a) = f(b)$ is with $a = b$.

Another term that appears frequently is this: We say f is *onto* (or surjective) if every $y \in Y$ is of the form $y = f(x)$ for (at least one) $x \in X$. That is to say, f is onto if the set of all its values (often called the *image* of f) is all of Y. When we were considering linear equations $A\mathbf{x} = \mathbf{b}$ in Chapter 1, we found constraint equations that $\mathbf{b}$ must satisfy in order for the equation to be consistent. Vectors $\mathbf{b}$ satisfying those constraint equations are in the image of μ_A. The mapping μ_A is onto precisely when there are no constraint equations for consistency.

Last, a function $f: X \to Y$ that is both one-to-one and onto is often called a one-to-one correspondence between X and Y (or a bijection). We saw in Section 5 of Chapter 1 that $\mu_A : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto precisely when A is nonsingular.

As we just saw in proving associativity of matrix multiplication, for an $m \times n$ matrix A and an $n \times p$ matrix B,

$$(AB)\mathbf{c} = A(B\mathbf{c})$$

for every vector $\mathbf{c} \in \mathbb{R}^p$. We can now rewrite this as

$$\mu_{AB}(\mathbf{c}) = \mu_A(\mu_B(\mathbf{c})) = \left(\mu_A \circ \mu_B\right)(\mathbf{c}),$$

where the latter notation denotes composition of functions. Of course, this formula is the real motivation for defining matrix multiplication as we did. In fact, one might define the matrix product as a composition of functions and then derive the computational formula. Now, we know that *composition of functions is associative* (even though it is not commutative):

$$(f\circ g)\circ h = f\circ(g\circ h),$$

from which we infer that

$$\begin{aligned}\left(\mu_A\circ\mu_B\right)\circ\mu_C &= \mu_A\circ\left(\mu_B\circ\mu_C\right), \quad \text{and so}\\ \mu_{(AB)C} &= \mu_{A(BC)}; \quad \text{that is,}\\ (AB)C &= A(BC).\end{aligned}$$

This is how one should understand matrix multiplication and its associativity.

Remark. Mathematicians will often express the rule $\mu_{AB} = \mu_A\circ\mu_B$ schematically by the following diagram:

$$\mathbb{R}^p \xrightarrow{\mu_B} \mathbb{R}^n \xrightarrow{\mu_A} \mathbb{R}^m \quad (\text{with } \mu_{AB}: \mathbb{R}^p \to \mathbb{R}^m)$$

We will continue to explore the interpretation of matrices as functions in the next section.

Exercises 2.1

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$. Calculate each of the following expressions or explain why it is not defined.

a. $A+B$ d. $C+D$ *g. AC j. DB

*b. $2A-B$ e. AB *h. CA *k. CD

c. $A-C$ *f. BA i. BD *l. DC

2. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$. Show that $AB = \mathrm{O}$ but $BA \neq \mathrm{O}$. Explain this result geometrically.

3. Prove Proposition 1.1. While you're at it, prove (using these properties) that for any $A \in \mathcal{M}_{m\times n}$, $0A = \mathrm{O}$.

4. a. Prove the remainder of Proposition 1.2.

b. Interpret parts **1**, **2**, and **3** of Proposition 1.2 in terms of properties of functions.

c. Suppose Charlie has carefully proved the first statement in part **2** and offers the following justification of the second: Since $(B+B')A = BA + B'A$, we now have $A(B+B') = (B+B')A = BA+B'A = AB+AB' = A(B+B')$. Decide whether he is correct.

5. *a. If A is an $m\times n$ matrix and $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, show that $A = \mathrm{O}$.

b. If A and B are $m\times n$ matrices and $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, show that $A = B$.

6. Prove or give a counterexample. Assume all the matrices are $n\times n$.

a. If $AB = CB$ and $B \neq \mathrm{O}$, then $A = C$.

b. If $A^2 = A$, then $A = \mathrm{O}$ or $A = I$.

c. $(A+B)(A-B)=A^2-B^2$.
d. If $AB=CB$ and B is nonsingular, then $A=C$.
e. If $AB=BC$ and B is nonsingular, then $A=C$.

In the box on p. 52, we suggested that in such a problem you might try $n=1$ to get intuition. Well, if we have real numbers a, b, and c satisfying $ab=cb$, then $ab-cb=(a-c)b=0$, so $b=0$ or $a=c$. Similarly, if $a^2=a$, then $a^2-a=a(a-1)=0$, so $a=0$ or $a=1$, and so on. So, once again, it's not clear that the case $n=1$ gives much insight into the general case. But it might lead us to the right question: Is it true for $n\times n$ matrices that $AB=O$ implies $A=O$ or $B=O$?

To answer this question, you might either play around with numerical examples (e.g., with 2×2 matrices) or interpret this matrix product geometrically: What does it say about the relation between the rows of A and the columns of B?

7. Find all 2×2 matrices $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

a. $A^2=I_2$ *b. $A^2=O$ c. $A^2=-I_2$

8. For each of the following matrices A, find a formula for A^k for positive integers k. (If you know how to do proof by induction, please do.)

a. $A=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ b. $A=\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ c. $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

9. (Block multiplication) We can think of an $(m+n)\times(m+n)$ matrix as being decomposed into "blocks," and thinking of these blocks as matrices themselves, we can form products and sums appropriately. Suppose A and A' are $m\times m$ matrices, B and B' are $m\times n$ matrices, C and C' are $n\times m$ matrices, and D and D' are $n\times n$ matrices. Verify the following formula for the product of "block" matrices:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]=\left[\begin{array}{c|c} AA'+BC' & AB'+BD' \\ \hline CA'+DC' & CB'+DD' \end{array}\right].$$

10. Suppose A and B are nonsingular $n\times n$ matrices. Prove that AB is nonsingular.

Although it is tempting to try to show that the reduced echelon form of AB is the identity matrix, there is no direct way to do this. As is the case in most non-numerical problems regarding nonsingularity, you should remember that AB is nonsingular precisely when the only solution of $(AB)\mathbf{x}=\mathbf{0}$ is $\mathbf{x}=\mathbf{0}$.

11. $^\sharp$a. Suppose $A\in\mathcal{M}_{m\times n}$, $B\in\mathcal{M}_{n\times m}$, and $BA=I_n$. Prove that if for some $\mathbf{b}\in\mathbb{R}^m$ the equation $A\mathbf{x}=\mathbf{b}$ has a solution, then that solution is unique.

b. Suppose $A\in\mathcal{M}_{m\times n}$, $C\in\mathcal{M}_{n\times m}$, and $AC=I_m$. Prove that the system $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b}\in\mathbb{R}^m$.

To show that *if* a solution exists, then it is unique, one approach (which works well here) is to *suppose* that $\mathbf{x}$ satisfies the equation and find a formula that determines it. Another approach is to assume that $\mathbf{x}$ and $\mathbf{y}$ are both solutions and then use the equations to prove that $\mathbf{x} = \mathbf{y}$.

To prove that a solution exists, the direct approach (which works here) is to find *some* $\mathbf{x}$ that works—even if that means guessing. A more subtle approach to existence questions involves proof by contradiction (see the box on p. 18): Assume there is no solution, and deduce from this assumption something that is known to be false.

$^\sharp$c. Suppose $A \in \mathcal{M}_{m\times n}$ and $B, C \in \mathcal{M}_{n\times m}$ are matrices that satisfy $BA = I_n$ and $AC = I_m$. Prove that $B = C$.

12. An $n \times n$ matrix is called a *permutation matrix* if it has a single 1 in each row and column and all its remaining entries are 0.

a. Write down all the 2×2 permutation matrices. How many are there?

b. Write down all the 3×3 permutation matrices. How many are there?

c. Show that the product of two permutation matrices is again a permutation matrix. Do they commute?

d. Prove that every permutation matrix is nonsingular.

e. If A is an $n \times n$ matrix and P is an $n \times n$ permutation matrix, describe the columns of AP and the rows of PA.

13. Find matrices A so that

a. $A \neq \mathrm{O}$, but $A^2 = \mathrm{O}$

b. $A^2 \neq \mathrm{O}$, but $A^3 = \mathrm{O}$

Can you make a conjecture about matrices satisfying $A^{n-1} \neq \mathrm{O}$ but $A^n = \mathrm{O}$?

14. Find all 2×2 matrices A that commute with all 2×2 matrices B. That is, if $AB = BA$ for all $B \in \mathcal{M}_{2\times 2}$, what are the possible matrices that A can be?

15. (The binomial theorem for matrices) Suppose A and B are $n \times n$ matrices with the property that $AB = BA$. Prove that for any positive integer k, we have

$$\begin{aligned}(A+B)^k &= \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} A^{k-i} B^i \\ &= A^k + kA^{k-1}B + \frac{k(k-1)}{2}A^{k-2}B^2 + \frac{k(k-1)(k-2)}{6}A^{k-3}B^3 \\ &\qquad + \cdots + kAB^{k-1} + B^k.\end{aligned}$$

Show that the result is false when $AB \neq BA$.

2 Linear Transformations: An Introduction

The function μ_A we defined at the end of Section 1 is a prototype of the functions one studies in linear algebra, called *linear transformations*. We shall explore them in greater detail in Chapter 4, but here we want to familiarize ourselves with a number of examples. First, a definition:

Definition. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* (or *linear map*) if it satisfies

(i) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(ii) $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and all scalars c.

These are often called the *linearity properties*.

EXAMPLE 1

Here are a few examples of functions, some linear, some not.

(a) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix}$. Let's decide whether it satisfies the two properties of a linear map.

(i)

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} (x_1 + y_1) + (x_2 + y_2) \\ (x_1 + y_1) \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) + (y_1 + y_2) \\ x_1 + y_1 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \end{aligned}$$

(ii)

$$\begin{aligned} T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + cx_2 \\ cx_1 \end{bmatrix} \\ &= c\begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix} = cT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \quad \text{for all scalars } c. \end{aligned}$$

Thus, T is a linear map.

It is important to remember that we have to check that the equation $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ holds for *all* vectors $\mathbf{x}$ and $\mathbf{y}$, so the argument must be an algebraic one using variables. Similarly, we must show $T(c\mathbf{x}) = cT(\mathbf{x})$ for all vectors $\mathbf{x}$ and all scalars c. It is not enough to check a few cases.

(b) What about the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2 \end{bmatrix}$? Here we can see that both properties fail, but we only need to provide evidence that *one* fails. For example, $T\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \neq 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which is what $3T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

would be. The reader can also try checking whether

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right).$$

> Just a reminder: To check that a multi-part (in this case, two-part) definition *holds*, we must check each condition. However, to show that a multi-part definition *fails*, we only need to show that *one* of the criteria does not hold.

(c) We learned in Section 2 of Chapter 1 to project one vector onto another. We now think of this as defining a function: Let $\mathbf{a} \in \mathbb{R}^2$ be fixed and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\mathbf{x}) = \text{proj}_{\mathbf{a}}\mathbf{x}$. One can give a geometric argument that this is a linear map (see Exercise 15), but we will use our earlier formula from p. 22 to establish this. Since

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a},$$

we have

(i) $T(\mathbf{x}+\mathbf{y}) = \dfrac{(\mathbf{x}+\mathbf{y}) \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \dfrac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} + \dfrac{\mathbf{y} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = T(\mathbf{x}) + T(\mathbf{y})$, and

(ii) $T(c\mathbf{x}) = \dfrac{(c\mathbf{x}) \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = c\dfrac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = cT(\mathbf{x})$.

Notice that if we replace $\mathbf{a}$ with a nonzero scalar multiple of $\mathbf{a}$, the map T doesn't change. For this reason, we will refer to $T = \text{proj}_{\mathbf{a}}$ as the *projection of* $\mathbb{R}^2$ *onto the line* ℓ, where ℓ is the line spanned by $\mathbf{a}$. We will denote this mapping by P_ℓ.

(d) It follows from Exercise 1.4.13 (see also Proposition 5.2 of Chapter 1) that for any $m \times n$ matrix A, the function $\mu_A: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. ▲

EXAMPLE 2

Expanding on the previous example, we consider the linear transformations $\mu_A: \mathbb{R}^2 \to \mathbb{R}^2$ for some specific 2×2 matrices A and give geometric interpretations of these maps.

(a) If $A = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = O$ is the zero matrix, then $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^2$, so μ_A sends every vector in $\mathbb{R}^2$ to the zero vector $\mathbf{0}$. If $B = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2$ is the 2×2 identity matrix, then $B\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$. The function μ_B is the *identity map* from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(b) Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by multiplication by the matrix

$$A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}.$$

The effect of T is pictured in Figure 2.1. One might slide a deck of cards in this fashion, and such a motion is called a *shear*.

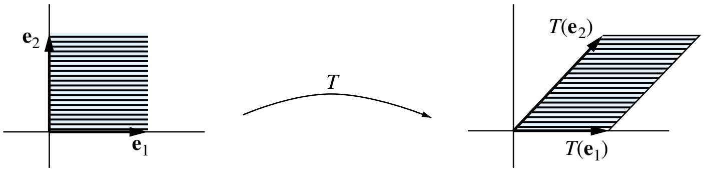


FIGURE 2.1

(c) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then we have

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix},$$

and we see in Figure 2.2 that $A\mathbf{x}$ is obtained by rotating $\mathbf{x}$ an angle of $\pi/2$ counterclockwise.

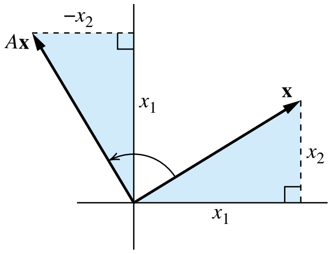


FIGURE 2.2

(d) Let

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we have

$$B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix},$$

as shown in Figure 2.3. We see that $B\mathbf{x}$ is the "mirror image" of the vector $\mathbf{x}$, reflecting across the "mirror" $x_1 = x_2$. In general, we say $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by *reflection* across a line ℓ if, for every $\mathbf{x} \in \mathbb{R}^2$, $T(\mathbf{x})$ has the same length as $\mathbf{x}$ and the two vectors make the same angle with ℓ.[1]

(e) Continuing with the matrices A and B from parts *c* and *d*, respectively, let's consider the function $\mu_{AB}: \mathbb{R}^2 \to \mathbb{R}^2$. Recalling that $\mu_{AB} = \mu_A \circ \mu_B$, we have the situation shown in Figure 2.4. The picture suggests that μ_{AB} is the linear transformation that gives reflection across the vertical axis, $x_1 = 0$. To be sure, we can compute algebraically:

$$AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

[1] Strictly speaking, if the angle from ℓ to $\mathbf{x}$ is θ, then the angle from ℓ to $T(\mathbf{x})$ should be $-\theta$.

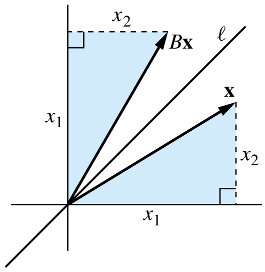


FIGURE 2.3

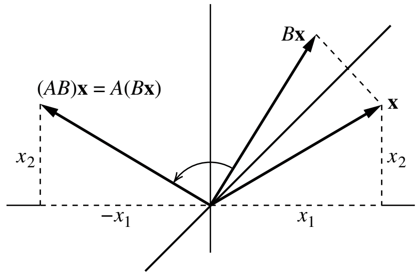


FIGURE 2.4

and so

$$(AB)\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}.$$

This is indeed the formula for the reflection across the vertical axis. So we have seen that the function $\mu_A \circ \mu_B$—the composition of the reflection about the line $x_1 = x_2$ and a rotation through an angle of $\pi/2$—is the reflection across the line $x_1 = 0$. On the

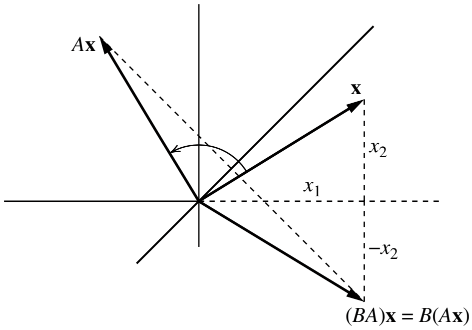


FIGURE 2.5

other hand, as indicated in Figure 2.5, the function $\mu_B \circ \mu_A = \mu_{BA} : \mathbb{R}^2 \to \mathbb{R}^2$, as we leave it to the reader to check, is the reflection across the line $x_2 = 0$. ▲

EXAMPLE 3

Continuing Example 2(d), if ℓ is a line in $\mathbb{R}^2$ through the origin, the *reflection across* ℓ is the map $R_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\mathbf{x}$ to its "mirror image" in ℓ. We begin by writing $\mathbf{x} = \mathbf{x}^{\parallel} + \mathbf{x}^{\perp}$, where $\mathbf{x}^{\parallel}$ is parallel to ℓ and $\mathbf{x}^{\perp}$ is orthogonal to ℓ, as in Section 2 of Chapter 1. Then, as we see in Figure 2.6,

$$R_\ell(\mathbf{x}) = \mathbf{x}^{\parallel} - \mathbf{x}^{\perp} = \mathbf{x}^{\parallel} - (\mathbf{x} - \mathbf{x}^{\parallel}) = 2\mathbf{x}^{\parallel} - \mathbf{x}.$$

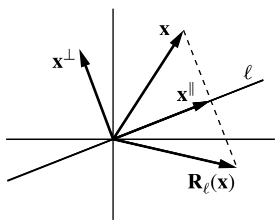


FIGURE 2.6

Using the notation of Example 1(c), we have $\mathbf{x}^{\parallel} = P_\ell(\mathbf{x})$, and so $R_\ell(\mathbf{x}) = 2P_\ell(\mathbf{x}) - \mathbf{x}$, or, in functional notation, $R_\ell = 2P_\ell - I$, where $I\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map. One can now use the result of Exercise 11 to deduce that R_ℓ is a linear map.

It is worth noting that $R_\ell(\mathbf{x})$ is the vector on the other side of ℓ from $\mathbf{x}$ that has the same length as $\mathbf{x}$ and makes the same angle with ℓ as $\mathbf{x}$ does. In particular, the right triangle with leg $\mathbf{x}^{\parallel}$ and hypotenuse $\mathbf{x}$ is congruent to the right triangle with leg $\mathbf{x}^{\parallel}$ and hypotenuse $R_\ell(\mathbf{x})$. This observation leads to a geometric argument that reflection across ℓ is indeed a linear transformation (see Exercise 15). ▲

EXAMPLE 4

We conclude this discussion with a few examples of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(a) Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because μ_A leaves the x_2x_3-plane fixed and sends $(1,0,0)$ to $(-1,0,0)$, we see that $A\mathbf{x}$ is obtained by reflecting $\mathbf{x}$ across the x_2x_3-plane.

(b) Let

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have

$$B\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \end{bmatrix}.$$

We see that μ_B leaves the x_3-axis fixed and rotates the x_1x_2-plane through an angle of $\pi/2$. Thus, μ_B rotates an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ an angle of $\pi/2$ about the x_3-axis, as pictured in Figure 2.7.

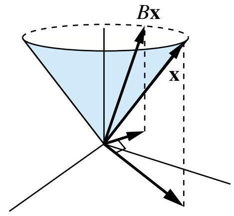


FIGURE 2.7

(c) Let $\mathbf{a} = (1,1,1)$. For any $\mathbf{x} \in \mathbb{R}^3$, we know that the projection of $\mathbf{x}$ onto $\mathbf{a}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{1}{3}(x_1+x_2+x_3)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x_1+x_2+x_3 \\ x_1+x_2+x_3 \\ x_1+x_2+x_3 \end{bmatrix}.$$

Thus, if we define the matrix

$$C = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

we have $\text{proj}_{\mathbf{a}}\mathbf{x} = C\mathbf{x}$. In particular, $\text{proj}_{\mathbf{a}}$ is the linear transformation μ_C. As we did earlier in $\mathbb{R}^2$, we can also denote this linear map by P_ℓ, where ℓ is the line spanned by $\mathbf{a}$. ▲

2.1 The Standard Matrix of a Linear Transformation

When we examine the previous examples, we find a geometric meaning of the *column* vectors of the matrices. As we know, when we multiply a matrix by a vector, we get the appropriate linear combination of the columns of the matrix. In particular,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

And so we see that the first column of A is the vector $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mu_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and the second column of A is the vector $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mu_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Turning this observation on its head, we note that we can find the matrix A (and hence the linear map μ_A) by finding the two vectors $\mu_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $\mu_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This seems surprising at first, as the function μ_A is completely determined by what it does to only two (nonparallel) vectors in $\mathbb{R}^2$.

This is, in fact, a general property of linear maps. If, for example, $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we write

$$\mathbf{x} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and so, by the linearity properties,

$$\begin{aligned} T(\mathbf{x}) &= T\left(x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right). \end{aligned}$$

That is, once we know the two vectors $\mathbf{v}_1 = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $\mathbf{v}_2 = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, we can determine $T(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^2$. Indeed, if we create a 2×2 matrix by inserting $\mathbf{v}_1$ as the first column and $\mathbf{v}_2$ as the second column, then it follows from what we've done that $T = \mu_A$. Specifically, if

$$\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix},$$

then we obtain

$$A = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

A is called *the standard matrix* for T. (This entire discussion works more generally for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, but we will postpone that to Chapter 4.)

WARNING In order to apply the procedure we have just outlined, one *must* know in advance that the given function T is *linear*. If it is not, the matrix A constructed in this manner will *not* reproduce the original function T.

EXAMPLE 5

Let ℓ be the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $P_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto ℓ. We checked in Example 1(c) that this is a linear map. Thus, we can find the standard matrix for P_ℓ. To do this, we compute

$$P\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad P\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{2}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so the standard matrix representing P_ℓ is

$$A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Since we know that reflection across ℓ is given by $R_\ell = 2P_\ell - I$, the standard matrix for R_ℓ will be

$$B = 2 \cdot \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}.$$

We ask the reader to find the matrix for reflection across a general line in Exercise 12. ▲

EXAMPLE 6

Generalizing Example 2(c), we consider the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We see that

$$A_\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad A_\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

as pictured in Figure 2.8. Thus, the function μ_{A_θ} rotates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ through the angle θ, and we strongly suspect that $\mu_{A_\theta}(\mathbf{x}) = A_\theta\mathbf{x}$ should be the vector obtained by rotating $\mathbf{x}$ through angle θ. We leave it to the reader to check in Exercise 8 that this is the case, and we call A_θ a *rotation matrix*.

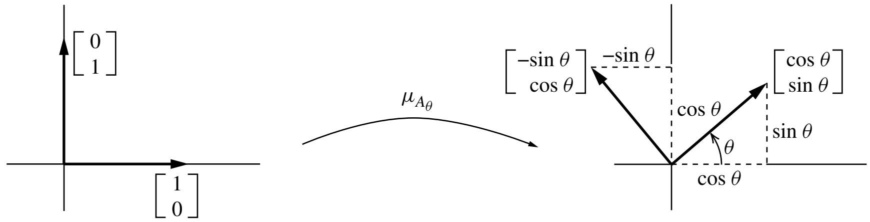


FIGURE 2.8

On the other hand, we could equally well have started with the map $T_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by rotating each vector counterclockwise by the angle θ. To take a geometric definition, the length of $T_\theta(\mathbf{x})$ is the same as the length of $\mathbf{x}$, and the angle between them is θ. Why is this map linear? Here is a detailed geometric justification. It is clear that if we rotate $\mathbf{x}$ and then multiply by a scalar c, we get the same result as rotating the vector $c\mathbf{x}$ (officially, the vector has the right length and makes the right angle with $c\mathbf{x}$). Now, as indicated in Figure 2.9, since the angle between $T_\theta(\mathbf{x})$ and $T_\theta(\mathbf{y})$ equals the angle between $\mathbf{x}$ and $\mathbf{y}$ (why?) and since lengths are preserved, it follows from the side-angle-side congruence theorem that the shaded triangles are congruent, and hence the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$ is congruent to the parallelogram spanned by $T_\theta(\mathbf{x})$ and $T_\theta(\mathbf{y})$. The angle between $T_\theta(\mathbf{x}) + T_\theta(\mathbf{y})$ and $T_\theta(\mathbf{x})$ is the same as the angle between $\mathbf{x} + \mathbf{y}$ and $\mathbf{x}$, so, by simple arithmetic (do it!), the angle between $T_\theta(\mathbf{x}) + T_\theta(\mathbf{y})$ and $\mathbf{x} + \mathbf{y}$ is θ. Again because the parallelograms are congruent, $T_\theta(\mathbf{x}) + T_\theta(\mathbf{y})$ has the same length as $\mathbf{x} + \mathbf{y}$, hence the same length as $T_\theta(\mathbf{x} + \mathbf{y})$, and so the vectors $T_\theta(\mathbf{x}) + T_\theta(\mathbf{y})$ and $T_\theta(\mathbf{x} + \mathbf{y})$ must be equal. Whew!

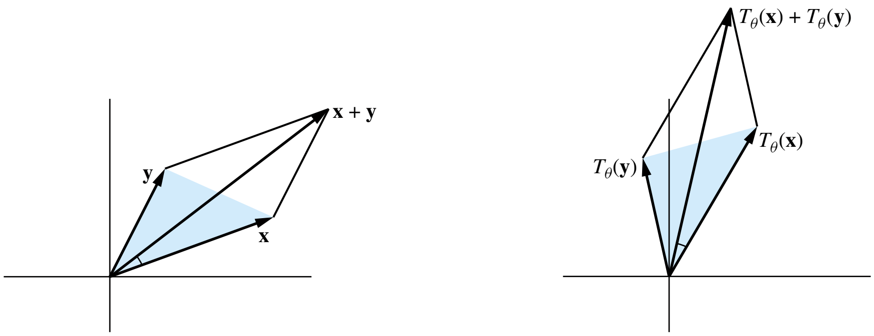


FIGURE 2.9

A natural question to ask is this: What is the product $A_\theta A_\phi$? The answer should be quite clear if we think of this as the composition of functions $\mu_{A_\theta A_\phi} = \mu_{A_\theta} \circ \mu_{A_\phi}$. We leave this to Exercise 7.

EXAMPLE 7

The geometric interpretation of a given linear transformation is not always easy to determine just by looking at the matrix. For example, if we let

$$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix},$$

then we might observe that for every $\mathbf{x} \in \mathbb{R}^2$, $A\mathbf{x}$ is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (why?). From our past experience, what does this suggest? As a clue to understanding the associated linear transformation, we might try calculating A^2, and we find that $A^2 = A$; it follows that $A^n = A$ for all positive integers n (why?). What is the geometric explanation?

With some care we can unravel the mystery:

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(x_1+2x_2) \\ \frac{2}{5}(x_1+2x_2) \end{bmatrix} = \frac{x_1+2x_2}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\mathbf{x}\cdot(1,2)}{\|(1,2)\|^2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is the projection of $\mathbf{x}$ onto the line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (Of course, if one remembers Example 5, this was really no mystery.) This explains why $A^2\mathbf{x} = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^2$: $A^2\mathbf{x} = A(A\mathbf{x})$, and once we've projected the vector $\mathbf{x}$ onto the line, it stays put. ▲

Exercises 2.2

1. Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Compute $T\left(2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}\right)$.

2. Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+2x_2+x_3 \\ 3x_1-x_2-x_3 \end{bmatrix}$. Find a matrix A so that $T = \mu_A$.

3. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. In each case, use the information provided to find the standard matrix A for T.

*a. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b. $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

c. $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

***4.** Determine whether each of the following functions is a linear transformation. If so, provide a proof; if not, explain why.

a. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+2x_2 \\ x_2^2 \end{bmatrix}$

b. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+2x_2 \\ 0 \end{bmatrix}$

c. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 - x_2$

d. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} |x_2| \\ 3x_1 \end{bmatrix}$

e. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + 3x_2 \end{bmatrix}$

f. $T\colon \mathbb{R}^n \to \mathbb{R}$ given by $T(\mathbf{x}) = \|\mathbf{x}\|$

5. Give 2×2 matrices A so that for any $\mathbf{x} \in \mathbb{R}^2$ we have, respectively:

a. $A\mathbf{x}$ is the vector whose components are, respectively, the sum and difference of the components of $\mathbf{x}$.

*b. $A\mathbf{x}$ is the vector obtained by projecting $\mathbf{x}$ onto the line $x_1 = x_2$ in $\mathbb{R}^2$.

c. $A\mathbf{x}$ is the vector obtained by first reflecting $\mathbf{x}$ across the line $x_1 = 0$ and then reflecting the resulting vector across the line $x_2 = x_1$.

d. $A\mathbf{x}$ is the vector obtained by projecting $\mathbf{x}$ onto the line $2x_1 - x_2 = 0$.

*e. $A\mathbf{x}$ is the vector obtained by first projecting $\mathbf{x}$ onto the line $2x_1 - x_2 = 0$ and then rotating the resulting vector $\pi/2$ counterclockwise.

f. $A\mathbf{x}$ is the vector obtained by first rotating $\mathbf{x}$ an angle of $\pi/2$ counterclockwise and then projecting the resulting vector onto the line $2x_1 - x_2 = 0$.

***6.** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by rotating the plane $\pi/2$ counterclockwise; let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by reflecting the plane across the line $x_1 + x_2 = 0$.

a. Give the standard matrices representing S and T.

b. Give the standard matrix representing $T \circ S$.

c. Give the standard matrix representing $S \circ T$.

7. a. Calculate $A_\theta A_\phi$ and $A_\phi A_\theta$. (Recall the definition of the rotation matrix on p. 98.)

b. Use your answer to part *a* to derive the addition formulas for sine and cosine.

8. Let A_θ be the rotation matrix defined on p. 98, $0 \le \theta \le \pi$. Prove that

a. $\|A_\theta \mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^2$.

b. the angle between $\mathbf{x}$ and $A_\theta \mathbf{x}$ is θ.

These properties characterize a rotation of the plane through angle θ.

9. Let ℓ be the line spanned by $\mathbf{a} \in \mathbb{R}^2$, and let $R_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by reflection across ℓ. Using the formula $R_\ell(\mathbf{x}) = \mathbf{x}^{\parallel} - \mathbf{x}^{\perp}$ given in Example 3, verify that

a. $\|R_\ell(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^2$.

b. $R_\ell(\mathbf{x}) \cdot \mathbf{a} = \mathbf{x} \cdot \mathbf{a}$ for all $\mathbf{x} \in \mathbb{R}^2$; i.e., the angle between $\mathbf{x}$ and ℓ is the same as the angle between $R_\ell(\mathbf{x})$ and ℓ.

10. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove the following:

a. $T(\mathbf{0}) = \mathbf{0}$

b. $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars a and b

11. a. Prove that if $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and c is any scalar, then the function $cT\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by $(cT)(\mathbf{x}) = cT(\mathbf{x})$ (i.e., the scalar c times the vector $T(\mathbf{x})$) is also a linear transformation.

b. Prove that if $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, then the function $S + T\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by $(S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x})$ is also a linear transformation.

c. Prove that if $S\colon \mathbb{R}^m \to \mathbb{R}^p$ and $T\colon \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, then the function $S \circ T\colon \mathbb{R}^n \to \mathbb{R}^p$ is also a linear transformation.

12. a. Let ℓ be the line spanned by $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$. Show that the standard matrix for R_ℓ is

$$R = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$

by using Figure 2.10 and basic geometry to find the reflections of (1, 0) and (0, 1).

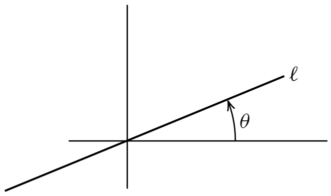


FIGURE 2.10

b. Derive this formula for R by using $R_\ell = 2P_\ell - I$ (see Example 3).

c. Letting A_θ be the rotation matrix defined on p. 98, check that

$$A_{2\theta}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = R = A_\theta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}A_{(-\theta)}.$$

d. Give geometric interpretations of these equalities.

13. Let ℓ be a line through the origin in $\mathbb{R}^2$.

a. Show that $P_\ell^2 = P_\ell \circ P_\ell = P_\ell$.

b. Show that $R_\ell^2 = R_\ell \circ R_\ell = I$.

14. Let ℓ_1 be the line through the origin in $\mathbb{R}^2$ making angle α with the x_1-axis, and let ℓ_2 be the line through the origin in $\mathbb{R}^2$ making angle β with the x_1-axis. Find $R_{\ell_2} \circ R_{\ell_1}$. (*Hint:* One approach is to use the matrix for reflection found in Exercise 12.)

15. Let $\ell \subset \mathbb{R}^2$ be a line through the origin.

a. Give a geometric argument that reflection across ℓ, the function $R_\ell : \mathbb{R}^2 \to \mathbb{R}^2$, is a linear transformation. (*Hint:* Consider the right triangles formed by $\mathbf{x}$ and $\mathbf{x}^{\parallel}$, $\mathbf{y}$ and $\mathbf{y}^{\parallel}$, and $\mathbf{x} + \mathbf{y}$ and $\mathbf{x}^{\parallel} + \mathbf{y}^{\parallel}$.)

b. Give a geometric argument that projection onto ℓ, the function $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$, is a linear transformation.

3 Inverse Matrices

Given an $m \times n$ matrix A, we are sometimes faced with the task of solving the equation $A\mathbf{x} = \mathbf{b}$ for several different values of $\mathbf{b} \in \mathbb{R}^m$. To accomplish this, it would be convenient to have an $n \times m$ matrix B satisfying $AB = I_m$: Taking $\mathbf{x} = B\mathbf{b}$, we will then have $A\mathbf{x} = A(B\mathbf{b}) = (AB)\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. This leads us to the following definition.

> ***Definition.*** Given an $m \times n$ matrix A, an $n \times m$ matrix B is called a *right inverse* of A if $AB = I_m$. Similarly, an $n \times m$ matrix C is called a *left inverse* of A if $CA = I_n$.

Note the symmetry here: If B is a right inverse of A, then A is a left inverse of B, and vice versa. Also, thinking in terms of linear transformations, if B is a right inverse of A, for example, then $\mu_A \circ \mu_B$ is the identity mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$.

EXAMPLE 1

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and so B is a right inverse of A (and A is a left inverse of B). Notice, however, that

$$BA = \begin{bmatrix} 0 & -1 \\ -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ -4 & 5 & -4 \\ -3 & 3 & -2 \end{bmatrix},$$

which is nothing like I_3. ▲

We observed earlier that if A has a right inverse, then we can always solve $A\mathbf{x} = \mathbf{b}$; i.e., this equation is consistent for every $\mathbf{b} \in \mathbb{R}^m$. On the other hand, if A has a left inverse, C, then a solution, *if it exists*, must be unique: If $A\mathbf{x} = \mathbf{b}$, then $C(A\mathbf{x}) = C\mathbf{b}$, and so $\mathbf{x} = I_n\mathbf{x} = (CA)\mathbf{x} = C(A\mathbf{x}) = C\mathbf{b}$. Thus, provided $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ must equal $C\mathbf{b}$, but maybe there aren't any solutions at all. To verify that $C\mathbf{b}$ is in fact a solution, we must calculate $A(C\mathbf{b})$ and see whether it is equal to $\mathbf{b}$. Of course, by associativity, this can be rewritten as $(AC)\mathbf{b} = \mathbf{b}$. This may or may not happen, but we do observe that if we want the vector $C\mathbf{b}$ to be a solution of $A\mathbf{x} = \mathbf{b}$ for *every* choice of $\mathbf{b} \in \mathbb{R}^m$, then we will need to have $AC = I_m$; i.e., we will need C to be both a left inverse *and* a right inverse of A. (This might be a good time to review the discussion of solving equations in the blue box on p. 23.)

We recall from Chapter 1 that, given the $m \times n$ matrix A, the equation $A\mathbf{x} = \mathbf{b}$ is *consistent* for all $\mathbf{b} \in \mathbb{R}^m$ precisely when the echelon form of A has no rows of 0's, i.e., when the rank of A is equal to m, the number of rows of A. On the other hand, the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution precisely when the rank of A is equal to n, the number of columns of A. Summarizing, we have the following proposition.

Proposition 3.1. *If the $m \times n$ matrix A has a right inverse, then the rank of A must be m, and if A has a left inverse, then its rank must be n. Thus, if A has both a left inverse and a right inverse, it must be square ($n \times n$) with rank n.*

Now suppose A is a square, $n \times n$, matrix with right inverse B and left inverse C, so that

$$AB = I_n = CA.$$

Then, exploiting associativity of matrix multiplication, we have

$$(*) \qquad C = CI_n = C(AB) = (CA)B = I_nB = B.$$

That is, if A has both a left inverse and a right inverse, they must be equal. This leads us to the following definition.

Definition. An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B satisfying

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

The matrix B is usually denoted A^{-1} (read "A inverse").

Remark. Note that if $B = A^{-1}$, then it is also the case that $A = B^{-1}$. We also note from equation $(*)$ that the inverse is unique: If B and C are both inverses of A, then, in particular, $AB = I_n$ and $CA = I_n$, so $B = C$.

EXAMPLE 2

Let

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Then $AB = I_2$ and $BA = I_2$, so B is the inverse matrix of A. ▲

It is a consequence of our earlier discussion that if A is an invertible $n \times n$ matrix, then $A\mathbf{x} = \mathbf{c}$ has a unique solution for every $\mathbf{c} \in \mathbb{R}^n$, and so it follows from Proposition 5.5 of Chapter 1 that A must be nonsingular. What about the converse? If A is nonsingular, must A be invertible? Well, if A is nonsingular, we know that every equation $A\mathbf{x} = \mathbf{c}$ has a unique solution. In particular, if $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector with all entries 0 except for a 1 in the j^{th} slot, there is a unique vector $\mathbf{b}_j$ that solves $A\mathbf{b}_j = \mathbf{e}_j$. If we let B be the $n \times n$ matrix whose column vectors are $\mathbf{b}_1, \dots, \mathbf{b}_n$, then we have

$$AB = A\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ | & | & & | \end{bmatrix} = I_n.$$

This suggests that the matrix we've constructed should be the inverse matrix of A. But we need to know that $BA = I_n$ as well. Here is a very elegant way to understand why this is so. We can find the matrix B by forming the giant augmented matrix (see Exercise 1.4.7)

$$\left[\begin{array}{c|ccc} & | & & | \\ A & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ & | & & | \end{array}\right] = \left[\begin{array}{c|c} A & I_n \end{array}\right]$$

and using Gaussian elimination to obtain the reduced echelon form

$$\left[\begin{array}{c|c} I_n & B \end{array}\right].$$

(Note that the reduced echelon form of A must be I_n because A is nonsingular.) Now here is the tricky part: By reversing the row operations, we find that the augmented matrix

$$\left[\begin{array}{c|c} B & I_n \end{array}\right]$$

is transformed to

$$\left[\begin{array}{c|c} I_n & A \end{array}\right].$$

This says that $BA = I_n$, which is what we needed to check. In conclusion, we have proved the following theorem.

Theorem 3.2. *An $n \times n$ matrix is nonsingular if and only if it is invertible.*

Note that Gaussian elimination will also let us know when A is not invertible: If we come to a row of 0's while we are reducing A to echelon form, then, of course, A is singular and so it cannot be invertible.

EXAMPLE 3

We wish to determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

(if it exists). We apply Gaussian elimination to the augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

Since we have determined that A is nonsingular, it follows that

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}.$$

(The reader should check our arithmetic by multiplying AA^{-1} or $A^{-1}A$.) ▲

EXAMPLE 4

It is convenient to derive a formula for the inverse (when it exists) of a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We assume $a \neq 0$ to start with. Then

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad (\text{assuming } ad - bc \neq 0)$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{b}{a}(-\frac{c}{ad-bc}) & -\frac{b}{a}\frac{a}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right],$$

and so we see that, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

As a check, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I_2 = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Of course, we have derived this assuming $a \neq 0$, but the reader can check easily that the formula works fine even when $a = 0$. We do see, however, from the row reduction that

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is nonsingular } \iff ad - bc \neq 0,}$$

because if $ad - bc = 0$, then we get a row of 0's in the echelon form of A. ▲

EXAMPLE 5

It follows immediately from Example 4 that for our rotation matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \text{we have} \quad A_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we see that this is the matrix $A_{(-\theta)}$. If we think about the corresponding functions μ_{A_θ} and $\mu_{A_{(-\theta)}}$, this result becomes obvious: To invert (or "undo") a rotation through angle θ, we must rotate through angle $-\theta$. ▲

By now it may have occurred to the reader that for square matrices, a one-sided inverse must actually be a true inverse. We formalize this observation here.

Corollary 3.3. *If A and B are $n \times n$ matrices satisfying $BA = I_n$, then $B = A^{-1}$ and $A = B^{-1}$.*

Proof. If $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = \mathbf{0}$, so, by Proposition 5.5 of Chapter 1, A is nonsingular. According to Theorem 3.2, A is therefore invertible. Since A has an inverse matrix, A^{-1}, we deduce that[2]

$$\begin{aligned}
BA &= I_n \\
&\Downarrow \text{ multiplying both sides of the equation by } A^{-1} \text{ on the right} \\
(BA)A^{-1} &= I_n A^{-1} \\
&\Downarrow \text{ using the associative property} \\
B(AA^{-1}) &= A^{-1} \\
&\Downarrow \text{ using the definition of } A^{-1} \\
B &= A^{-1},
\end{aligned}$$

as desired. Because $AB = I_n$ and $BA = I_n$, it now follows that $A = B^{-1}$, as well. □

EXAMPLE 6

We can use Gaussian elimination to find a *right* inverse of an $m \times n$ matrix A, so long as the rank of A is equal to m. The fact that we have free variables when $m < n$ will give many choices of right inverse. For example, taking

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix},$$

we apply Gaussian elimination to the augmented matrix

$$\left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|cc} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -2 & -2 & 1 \end{array}\right].$$

From this we see that the general solution of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

and the general solution of $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

If we take $s = t = 0$, we get the right inverse

$$B = \begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 0 & 0 \end{bmatrix},$$

[2] We are writing the "implies" symbol ($\Longrightarrow$) vertically so that we can indicate the reasoning in each step.

but we could take, say, $s = 1$ and $t = -1$ to obtain another right inverse,

$$B' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Finding a left inverse is a bit trickier. You can sometimes do it with a little guesswork, or you can set up a large system of equations to solve (thinking of the entries of the left inverse as the unknowns), but we will discuss a more systematic approach in the next section. ▲

We end this discussion with a very important observation.

Proposition 3.4. *Suppose A and B are invertible $n \times n$ matrices. Then their product AB is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Remark. Some people refer to this result rather endearingly as the "shoe-sock theorem," for to undo (invert) the process of putting on one's socks and then one's shoes, one must first remove the shoes and then remove the socks.

Proof. To prove the matrix AB is invertible, we need only check that the candidate for the inverse works. That is, we need to check that

$$(AB)(B^{-1}A^{-1}) = I_n \quad \text{and} \quad (B^{-1}A^{-1})(AB) = I_n.$$

But these follow immediately from associativity:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n, \quad \text{and}$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$ □

Exercises 2.3

1. Use Gaussian elimination to find A^{-1} (if it exists):

*a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

*e. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

*g. $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

2. In each case, given A and $\mathbf{b}$,

(i) Find A^{-1}.

(ii) Use your answer to (i) to solve $A\mathbf{x} = \mathbf{b}$.

(iii) Use your answer to (ii) to express $\mathbf{b}$ as a linear combination of the columns of A.

a. $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

*b. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

*d. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

***3.** Suppose A is an $n \times n$ matrix and B is an invertible $n \times n$ matrix. Simplify the following.

a. $(BAB^{-1})^2$

b. $(BAB^{-1})^n$ (n a positive integer)

c. $(BAB^{-1})^{-1}$ (what additional assumption is required here?)

***4.** Suppose A is an invertible $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = 7\mathbf{x}$. Calculate $A^{-1}\mathbf{x}$.

5. If P is a permutation matrix (see Exercise 2.1.12 for the definition), show that P is invertible and find P^{-1}.

6. a. Give another right inverse of the matrix A in Example 6.

b. Find two right inverses of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

c. Find two right inverses of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

7. a. Give a matrix that has a left inverse but no right inverse.

b. Give a matrix that has a right inverse but no left inverse.

c. Find two left inverses of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$.

***8.** Suppose A is a square matrix satisfying the equation $A^3 - 3A + I = \mathrm{O}$. Show that A is invertible. (*Hint:* Can you give an explicit formula for A^{-1}?)

9. Suppose A is a square matrix satisfying the equation $A^3 - 2I = \mathrm{O}$. Prove that A and $A - I$ are both invertible. (*Hint:* Give explicit formulas for their inverses. In the second case, a little trickery will be necessary: Start by factoring $x^3 - 1$.)

10. Suppose A is an $n \times n$ matrix with the property that $A - I$ is invertible.

a. For any $k = 1, 2, 3, \dots,$ give a formula for $(A - I)^{-1}(A^{k+1} - I)$. (*Hint:* Think about simplifying $\dfrac{x^{k+1} - 1}{x - 1}$ for $x \neq 1$.)

b. Use your answer to part *a* to find the number of paths of length ≤ 6 from node 1 to node 3 in Example 5 in Section 1.

11. Suppose A and B are $n \times n$ matrices. Prove that if AB is nonsingular, then both A and B are nonsingular. (*Hint:* First show that B is nonsingular; then use Theorem 3.2 and Proposition 3.4.)

12. Suppose A is an invertible $m \times m$ matrix and B is an invertible $n \times n$ matrix. (See Exercise 2.1.9 for the notion of block multiplication.)

a. Show that the matrix

$$\left[\begin{array}{c|c} A & \mathrm{O} \\ \hline \mathrm{O} & B \end{array}\right]$$

is invertible and give a formula for its inverse.

b. Suppose C is an arbitrary $m \times n$ matrix. Is the matrix

$$\left[\begin{array}{c|c} A & C \\ \hline O & B \end{array}\right]$$

invertible?

13. Suppose A is an invertible matrix and A^{-1} is known.

a. Suppose B is obtained from A by switching two columns. How can we find B^{-1} from A^{-1}? (*Hint:* Since $A^{-1}A = I$, we know the dot products of the *rows* of A^{-1} with the *columns* of A. So rearranging the columns of A to make B, we should be able to suitably rearrange the rows of A^{-1} to make B^{-1}.)

b. Suppose B is obtained from A by multiplying the j^{th} column by a nonzero scalar. How can we find B^{-1} from A^{-1}?

c. Suppose B is obtained from A by adding a scalar multiple of one column to another. How can we find B^{-1} from A^{-1}?

d. Suppose B is obtained from A by replacing the j^{th} column by a different vector. Assuming B is still invertible, how can we find B^{-1} from A^{-1}?

14. Let A be an $m \times n$ matrix.

a. Assume the rank of A is m and B is a right inverse of A. Show that B' is another right inverse of A if and only if $A(B - B') = O$ and that this occurs if and only if every column of $B - B'$ is orthogonal to every row of A.

b. Assume the rank of A is n and C is a left inverse of A. Show that C' is another left inverse of A if and only if $(C - C')A = O$ and that this occurs if and only if every row of $C - C'$ is orthogonal to every column of A.

15. Suppose A is an $m \times n$ matrix with a *unique* right inverse B. Prove that $m = n$ and that A is invertible.

16. Suppose A is an $n \times n$ matrix satisfying $A^{10} = O$. Prove that the matrix $I_n - A$ is invertible. (*Hint:* As a warm-up, try assuming $A^2 = O$.)

4 Elementary Matrices: Rows Get Equal Time

So far we have focused on interpreting matrix multiplication in terms of columns—that is, on the fact that the j^{th} column of AB is the product of A with the j^{th} column vector of B. But equally relevant is the following observation:

The i^{th} row of AB is the product of the i^{th} row vector of A with B.

Just as multiplying the matrix A by a column vector $\mathbf{x}$ on the right,

$$\begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

gives us the linear combination $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$ of the columns of A, the reader can easily check that multiplying A on the *left* by the row vector $[x_1\ x_2\ \cdots\ x_m]$,

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \text{——} & \mathbf{A}_1 & \text{——} \\ \text{——} & \mathbf{A}_2 & \text{——} \\ & \vdots & \\ \text{——} & \mathbf{A}_m & \text{——} \end{bmatrix},$$

yields the linear combination $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_m\mathbf{A}_m$ of the *rows* of A.

It should come as no surprise, then, that we can perform row operations on a matrix A by multiplying on the *left* by appropriately chosen matrices. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 4 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix},$$

then

$$E_1A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}, \quad E_2A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 20 & 24 \end{bmatrix}, \quad \text{and} \quad E_3A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}.$$

Here we establish the custom that when it is clearer to do so, we indicate 0 entries in a matrix by blank spaces.

Such matrices that give corresponding elementary row operations are called *elementary matrices*. Note that each elementary matrix differs from the identity matrix only in a small way.

(i) To interchange rows i and j, we should multiply by an elementary matrix of the form

$$\begin{array}{c} \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \\ \end{array} \begin{array}{c} \begin{matrix} & & \overset{i}{\downarrow} & & \overset{j}{\downarrow} & & \end{matrix} \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & \cdots & 0 & \cdots & 1 & \cdots & \\ & & & \ddots & & & \\ & \cdots & 1 & \cdots & 0 & \cdots & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}.$$

(ii) To multiply row i by a scalar c, we should multiply by an elementary matrix of the form

$$\begin{array}{c} \\ \\ \\ \\ i \rightarrow \\ \\ \\ \\ \end{array}\begin{array}{c} \qquad\qquad\quad i \\ \qquad\qquad\quad \downarrow \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}.$$

(iii) To add c times row i to row j, we should multiply by an elementary matrix of the form

$$\begin{array}{c} \\ \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \\ \end{array}\begin{array}{c} \qquad\quad i \qquad\quad j \\ \qquad\quad \downarrow \qquad\quad \downarrow \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & \cdots & c & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array}.$$

Here's an easy way to remember the form of these matrices: *Each elementary matrix is obtained by performing the corresponding elementary row operation on the identity matrix.*

EXAMPLE 1

Let $A = \begin{bmatrix} 4 & 3 & 5 \\ 1 & 2 & 5 \end{bmatrix}$. We put A in reduced echelon form by the following sequence of row operations:

$$\begin{bmatrix} 4 & 3 & 5 \\ 1 & 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -15 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$$

These steps correspond to multiplying, in sequence from right to left, by the elementary matrices

$$E_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & \\ & -\frac{1}{5} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -2 \\ & 1 \end{bmatrix}.$$

Now the reader can check that

$$E = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -2 \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & \\ -4 & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

and, indeed,

$$EA = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 4 & 3 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix},$$

as it should. Remember: The elementary matrices are arranged *from right to left* in the order in which the operations are done on A. ▲

EXAMPLE 2

Let's revisit Example 6 on p. 47. Let

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \end{bmatrix}.$$

To clear out the entries below the first pivot, we must multiply by the product of the two elementary matrices E_1 and E_2:

$$E_2E_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -2 & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ -2 & & & 1 \end{bmatrix};$$

to change the pivot in the second row to 1 and then clear out below, we multiply first by

$$E_3 = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

and then by the product

$$E_5E_4 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 3 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & 3 & & 1 \end{bmatrix}.$$

We next change the pivot in the third row to 1 and clear out below, multiplying by

$$E_6 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{2} & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad E_7 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -3 & 1 \end{bmatrix}.$$

Now we clear out above the pivots by multiplying by

$$E_8 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad E_9 = \begin{bmatrix} 1 & -1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The net result is this: When we multiply the product

$$E_9E_8E_7E_6(E_5E_4)E_3(E_2E_1) = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{9}{4} & -\frac{3}{2} & 1 \end{bmatrix}$$

by the original matrix, we do in fact get the reduced echelon form:

$$\begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{9}{4} & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ ▲

We now turn to some applications of elementary matrices to concepts we have studied earlier. Recall from Chapter 1 that if we want to find the constraint equations that a vector $\mathbf{b}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to be consistent, we reduce the augmented matrix $[\,A \mid \mathbf{b}\,]$ to echelon form $[\,U \mid \mathbf{c}\,]$ and set equal to 0 those entries of $\mathbf{c}$ corresponding to the rows of 0's in U. That is, when A is an $m \times n$ matrix of rank r, the constraint equations are merely the equations $c_{r+1} = \cdots = c_m = 0$. Letting E be the product of the elementary matrices corresponding to the elementary row operations required to put A in echelon form, we have $U = EA$, and so

$$(\dagger) \qquad [\,U \mid \mathbf{c}\,] = [\,EA \mid E\mathbf{b}\,].$$

That is, the constraint equations are the equations

$$\mathbf{E}_{r+1} \cdot \mathbf{b} = 0, \quad \ldots, \quad \mathbf{E}_m \cdot \mathbf{b} = 0,$$

where, we recall, $\mathbf{E}_{r+1}, \ldots, \mathbf{E}_m$ are the last $m - r$ row vectors of E. Interestingly, we can use the equation (†) to find a simple way to compute E: When we reduce the augmented matrix $[\,A \mid \mathbf{b}\,]$ to echelon form $[\,U \mid \mathbf{c}\,]$, E is the matrix satisfying $E\mathbf{b} = \mathbf{c}$.

EXAMPLE 3

Let's once again consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 & 0 \\ -1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 2 & -1 & 0 & 1 & -6 \end{bmatrix}$$

from Example 2, and let's find the constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent. We start with the augmented matrix

$$[A \mid \mathbf{b}] = \left[\begin{array}{rrrrr|c} 1 & 1 & 3 & -1 & 0 & b_1 \\ -1 & 1 & 1 & 1 & 2 & b_2 \\ 0 & 1 & 2 & 2 & -1 & b_3 \\ 2 & -1 & 0 & 1 & -6 & b_4 \end{array}\right]$$

and reduce to echelon form

$$[U \mid \mathbf{c}] = \left[\begin{array}{rrrrr|l} 1 & 1 & 3 & -1 & 0 & b_1 \\ 0 & 2 & 4 & 0 & 2 & b_1 + b_2 \\ 0 & 0 & 0 & 2 & -2 & -\frac{1}{2}b_1 - \frac{1}{2}b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 9b_2 - 6b_3 + 4b_4 \end{array}\right].$$

(Note that we have arranged to remove fractions from the entry in the last row.) Now it is easy to see that if

$$E\mathbf{b} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ -\frac{1}{2}b_1 - \frac{1}{2}b_2 + b_3 \\ b_1 + 9b_2 - 6b_3 + 4b_4 \end{bmatrix}, \quad \text{then} \quad E = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \\ 1 & 9 & -6 & 4 \end{bmatrix}.$$

The reader should check that, in fact, $EA = U$.

We could continue our Gaussian elimination to reach reduced echelon form:

$$[R \mid \mathbf{d}] = \left[\begin{array}{rrrrr|l} 1 & 0 & 1 & 0 & -2 & \frac{1}{4}b_1 - \frac{3}{4}b_2 + \frac{1}{2}b_3 \\ 0 & 1 & 2 & 0 & 1 & \frac{1}{2}b_1 + \frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & -1 & -\frac{1}{4}b_1 - \frac{1}{4}b_2 + \frac{1}{2}b_3 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 9b_2 - 6b_3 + 4b_4 \end{array}\right].$$

From this we see that $R = E'A$, where

$$E' = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 1 & 9 & -6 & 4 \end{bmatrix},$$

which is very close to—but not the same as—the product of elementary matrices we obtained at the end of Example 2. Can you explain why the first three rows must agree here, but not the last? ▲

EXAMPLE 4

If an $m \times n$ matrix A has rank n, then every column is a pivot column, so its reduced echelon form must be $R = \begin{bmatrix} I_n \\ \hline \mathrm{O} \end{bmatrix}$. If we find a product, E, of elementary matrices so that $EA = R$, then the first m rows of E will give us a left inverse of A. For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$, then we can take

$$E = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix},$$

and so

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

is a left inverse of A (as the diligent reader should check). ▲

4.1 The *LU* Decomposition

As a final topic in this section, let's reexamine the process of putting a matrix in echelon form by using elementary matrices. The crucial point is that elementary matrices are invertible and their inverses are elementary matrices of the same type (see Exercise 7). If $E = E_k \cdots E_2 E_1$ is the product of the elementary matrices we use to reduce A to echelon form, then $U = EA$ and so $A = E^{-1}U$. Suppose that we use *only* lower triangular elementary matrices of type **(iii)**: No row interchanges are required, and no rows are multiplied through by a scalar. In this event, all the E_i are lower triangular matrices with 1's on the diagonal, and so E is lower triangular with 1's on the diagonal, and E^{-1} has the same property. In this case, then, we've written $A = LU$, where $L = E^{-1}$ is a lower triangular (square) matrix with 1's on the diagonal. This is called the LU *decomposition* of A.

EXAMPLE 5

Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 7 & 4 & 2 \\ -1 & 4 & 13 & -1 \end{bmatrix}.$$

We reduce A to echelon form by the following sequence of row operations:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 7 & 4 & 2 \\ -1 & 4 & 13 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ -1 & 4 & 13 & -1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 6 & 12 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

This is accomplished by multiplying by the respective elementary matrices

$$E_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}.$$

Thus we have the equation

$$E_3 E_2 E_1 A = U,$$

whence

$$A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U.$$

Note that it is easier to calculate the inverses of the elementary matrices (see Exercise 7) and then calculate their product. In our case,

$$E_1^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}, \quad \text{and} \quad E_3^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix},$$

and so

$$\begin{aligned} L = E_1^{-1} E_2^{-1} E_3^{-1} &= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & 2 & 1 \end{bmatrix}. \end{aligned}$$

In fact, we see that when $i > j$, the ij-entry of L is the *negative* of the multiple of row j that we added to row i during our row operations.

Our LU decomposition, then, is as follows:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 7 & 4 & 2 \\ -1 & 4 & 13 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = LU.$$

▲

EXAMPLE 6

We reiterate that the LU decomposition exists only when no row interchanges are required to reduce the matrix to echelon form. For example, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no such expression. See Exercise 14. ▲

We shall see in Chapter 3 that, given the LU decomposition of a matrix A, we can read off a great deal of information. But the main reason it is of interest is this: To solve $A\mathbf{x} = \mathbf{b}$ for different vectors $\mathbf{b}$ using computers, it is significantly more cost-effective to use the LU decomposition (see Exercise 13). Notice that $A\mathbf{x} = \mathbf{b}$ if and only if $(LU)\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$, so first we solve $L\mathbf{y} = \mathbf{b}$ (by "forward substitution") and then we solve $U\mathbf{x} = \mathbf{y}$ (by "back substitution"). Actually, working by hand, it is even easier to determine L^{-1}, which is the product of elementary matrices that puts A in echelon form ($L^{-1}A = U$), so then we find $\mathbf{y} = L^{-1}\mathbf{b}$ and solve $U\mathbf{x} = \mathbf{y}$ as before.

Exercises 2.4

*1. For each of the matrices A in Exercise 1.4.3, find a product of elementary matrices $E = \cdots E_2 E_1$ so that EA is in echelon form. Use the matrix E you've found to give constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent.

*2. For each of the matrices A in Exercise 1.4.3, use the method of Example 3 to find a matrix E so that $EA = U$, where U is in echelon form.

*3. Give the LU decomposition (when it exists) of each of the matrices A in Exercise 1.4.3.

*4. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 1 \\ 2 & 1 & 0 & 0 & 5 \end{bmatrix}$.

a. Give the LU decomposition of A.

b. Give the reduced echelon form of A.

5. Find a left inverse of each of the following matrices A using the method of Example 4.

a. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ & 3 & 1 \\ & & -1 \end{bmatrix}$, solve $A\mathbf{x} = \mathbf{b}$, where

*a. $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ b. $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ *c. $\mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$

7. Show that the inverse of every elementary matrix is again an elementary matrix. Indeed, give a simple prescription for determining the inverse of each type of elementary matrix. (See the proof of Theorem 4.1 of Chapter 1.)

8. Prove or give a counterexample: Every invertible matrix can be written as a product of elementary matrices.

9. Use elementary matrices to prove Theorem 4.1 of Chapter 1.

10. *a. Suppose E_1 and E_2 are elementary matrices that correspond to adding multiples of the *same* row to other rows. Show that $E_1 E_2 = E_2 E_1$ and give a simple description of the product. Explain how to use this observation to compute the LU decomposition more efficiently.

b. In a similar vein, let $i < j$, $i < k$, and $j < \ell$. Let E_1 be an elementary matrix corresponding to adding a multiple of row i to row k, and let E_2 be an elementary matrix corresponding to adding a multiple of row j to row ℓ. Give a simple description of the product $E_1 E_2$, and explain how to use this observation to compute the LU decomposition more efficiently. Does $E_2 E_1 = E_1 E_2$ this time?

11. Complete the following alternative argument that the matrix obtained by Gaussian elimination must be the inverse matrix of A. It thereby provides another proof of Corollary 3.3. Suppose A is nonsingular.

a. Show that there are finitely many elementary matrices $E_1, E_2, \dots, E_k$ so that $E_k E_{k-1} \cdots E_2 E_1 A = I$.

b. Let $B = E_k E_{k-1} \cdots E_2 E_1$. Apply Proposition 3.4 to show that $A = B^{-1}$ and, thus, that $AB = I$.

12. Assume A and B are two $m \times n$ matrices with the same reduced echelon form. Show that there exists an invertible matrix E so that $EA = B$. Is the converse true?

13. We saw in Exercise 1.4.17 that it takes on the order of $n^3/3$ multiplications to put an $n \times n$ matrix in reduced echelon form (and, hence, to solve a square inhomogeneous system $A\mathbf{x} = \mathbf{b}$). Indeed, in solving that exercise, one shows that it takes on the order of $n^3/3$ multiplications to obtain U (and one obtains L just by bookkeeping). Show now that if one has different vectors $\mathbf{b}$ for which one wishes to solve $A\mathbf{x} = \mathbf{b}$, once one has $A = LU$, it takes on the order of n^2 multiplications to solve for $\mathbf{x}$ each time.

14. a. Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no LU decomposition.

b. Show that for any $m \times n$ matrix A, there is an $m \times m$ permutation matrix P so that PA does have an LU decomposition.

5 The Transpose

The final matrix operation we discuss in this chapter is the *transpose*. When A is an $m \times n$ matrix with entries a_{ij}, the matrix A^{T} (read "A transpose") is the $n \times m$ matrix whose ij-entry is a_{ji}; in other words, the i^{th} *row* of A^{T} is the i^{th} *column* of A. We say a square matrix A is *symmetric* if $A^{\mathsf{T}} = A$ and is *skew-symmetric* if $A^{\mathsf{T}} = -A$.

EXAMPLE 1

Suppose

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}.$$

Then $A^{\mathsf{T}} = B$, $B^{\mathsf{T}} = A$, $C^{\mathsf{T}} = D$, and $D^{\mathsf{T}} = C$. Note, in particular, that the transpose of a column vector, i.e., an $n \times 1$ matrix, is a row vector, i.e., a $1 \times n$ matrix. An example of a symmetric matrix is

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 7 \end{bmatrix}, \quad \text{since} \quad S^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 7 \end{bmatrix} = S.$$

▲

The basic properties of the transpose operation are as follows:

Proposition 5.1. *Let A and A' be $m \times n$ matrices, let B be an $n \times p$ matrix, and let c be a scalar. Then*

1. $(A^{\mathsf{T}})^{\mathsf{T}} = A$.
2. $(cA)^{\mathsf{T}} = cA^{\mathsf{T}}$.
3. $(A + A')^{\mathsf{T}} = A^{\mathsf{T}} + A'^{\mathsf{T}}$.
4. $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.
5. *When A is invertible, then so is A^{T}, and $(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}$.*

Proof. The first is obvious, inasmuch as we swap rows and columns and then swap again, returning to our original matrix. The second and third are immediate to check. The fourth result is more interesting, and we will use it to derive a crucial result in a moment. To prove **4**, note, first, that AB is an $m \times p$ matrix, so $(AB)^{\mathsf{T}}$ will be a $p \times m$ matrix; $B^{\mathsf{T}} A^{\mathsf{T}}$ is the product of a $p \times n$ matrix and an $n \times m$ matrix and hence will be $p \times m$ as well, so the shapes agree. Now, the ji-entry of AB is the dot product of the j^{th} row vector of A and the i^{th} column vector of B, i.e., the ij-entry of $(AB)^{\mathsf{T}}$ is

$$\big((AB)^{\mathsf{T}}\big)_{ij} = (AB)_{ji} = \mathbf{A}_j \cdot \mathbf{b}_i .$$

On the other hand, the ij-entry of $B^{\mathsf{T}} A^{\mathsf{T}}$ is the dot product of the i^{th} row vector of B^{T} and the j^{th} column vector of A^{T}; but this is, by definition, the dot product of the i^{th} column vector of B and the j^{th} row vector of A. That is,

$$(B^{\mathsf{T}} A^{\mathsf{T}})_{ij} = \mathbf{b}_i \cdot \mathbf{A}_j ,$$

and, since dot product is commutative, the two formulas agree. The proof of **5** is left to Exercise 8. □

The transpose matrix will be important to us because of the interplay between dot product and transpose. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then by virtue of our very definition of matrix multiplication,

$$\boxed{\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathsf{T}} \mathbf{y},}$$

provided we agree to think of a 1×1 matrix as a scalar. (On the right-hand side we are multiplying a $1 \times n$ matrix by an $n \times 1$ matrix.) Now we have this highly useful proposition:

Proposition 5.2. *Let A be an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{y} \in \mathbb{R}^m$. Then*

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^{\mathsf{T}} \mathbf{y}.$$

(On the left, we take the dot product of vectors in $\mathbb{R}^m$; on the right, of vectors in $\mathbb{R}^n$.)

Remark. You might remember this: To move the matrix "across the dot product," you must transpose it.

Proof. We just calculate, using the formula for the transpose of a product and, as usual, associativity:

$$A\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x})^{\mathsf{T}} \mathbf{y} = (\mathbf{x}^{\mathsf{T}} A^{\mathsf{T}}) \mathbf{y} = \mathbf{x}^{\mathsf{T}} (A^{\mathsf{T}} \mathbf{y}) = \mathbf{x} \cdot A^{\mathsf{T}} \mathbf{y}.$$ □

EXAMPLE 2

We return to the economic interpretation of dot product given in the Remark on p. 25. Suppose that m different ingredients are required to manufacture n different products. To manufacture the product vector $\mathbf{x} = (x_1, \dots, x_n)$ requires the ingredient vector $\mathbf{y} =$

$(y_1, \dots, y_m)$, and we suppose that $\mathbf{x}$ and $\mathbf{y}$ are related by the equation $\mathbf{y} = A\mathbf{x}$ for some $m \times n$ matrix A. If each unit of ingredient j costs a price p_j, then the cost of producing $\mathbf{x}$ is

$$\sum_{j=1}^{m} p_j y_j = \mathbf{y} \cdot \mathbf{p} = A\mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot A^{\mathsf{T}}\mathbf{p} = \sum_{i=1}^{n} q_i x_i,$$

where $\mathbf{q} = A^{\mathsf{T}}\mathbf{p}$. Notice then that q_i is the amount it costs to produce a unit of the i^{th} product. Our fundamental formula, Proposition 5.2, tells us that the total cost of the ingredients should equal the total worth of the products we manufacture. See Exercise 18 for a less abstract (but more fattening) example. ▲

EXAMPLE 3

We just saw that when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the matrix product $\mathbf{x}^{\mathsf{T}}\mathbf{y}$ is a 1×1 matrix. However, when we switch the position of the transpose and calculate $\mathbf{x}\mathbf{y}^{\mathsf{T}}$, the result is an $n \times n$ matrix (see Exercise 13). A particularly important application of this has arisen already in Chapter 1. Given a vector $\mathbf{a} \in \mathbb{R}^n$, consider the $n \times n$ matrix $A = \mathbf{a}\mathbf{a}^{\mathsf{T}}$. What does it mean? That is, what is the associated linear transformation μ_A? Well, by the associativity of multiplication, we have $A\mathbf{x} = (\mathbf{a}\mathbf{a}^{\mathsf{T}})\mathbf{x} = \mathbf{a}(\mathbf{a}^{\mathsf{T}}\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})\mathbf{a}$. When $\mathbf{a}$ is a unit vector, this is the projection of $\mathbf{x}$ onto $\mathbf{a}$. And, in general, we can now write

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{x}}{\|\mathbf{a}\|^2}\mathbf{a} = \left(\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^{\mathsf{T}}\right)\mathbf{x}.$$

We will see the importance of this formulation in Chapter 4. ▲

Exercises 2.5

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$. Calculate each of the following expressions or explain why it is not defined.

a. A^{T} d. $C^{\mathsf{T}} + D$ *g. $C^{\mathsf{T}}A^{\mathsf{T}}$ *j. CC^{T}
*b. $2A - B^{\mathsf{T}}$ *e. $A^{\mathsf{T}}C$ h. BD^{T} *k. $C^{\mathsf{T}}C$
c. C^{T} f. AC^{T} i. $D^{\mathsf{T}}B$ l. $C^{\mathsf{T}}D^{\mathsf{T}}$

2. Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$. Calculate the following matrices.

*a. $\mathbf{a}\mathbf{a}^{\mathsf{T}}$ c. $\mathbf{b}^{\mathsf{T}}\mathbf{b}$ e. $\mathbf{a}\mathbf{b}^{\mathsf{T}}$ g. $\mathbf{b}^{\mathsf{T}}\mathbf{a}$
*b. $\mathbf{a}^{\mathsf{T}}\mathbf{a}$ d. $\mathbf{b}\mathbf{b}^{\mathsf{T}}$ *f. $\mathbf{a}^{\mathsf{T}}\mathbf{b}$ h. $\mathbf{b}\mathbf{a}^{\mathsf{T}}$

3. Following Example 3, find the standard matrix for the projection $\text{proj}_{\mathbf{a}}$.

a. $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *b. $\mathbf{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ c. $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ d. $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

♯**4.** Suppose $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d} \in \mathbb{R}^n$. Check that, surprisingly,

$$\begin{bmatrix} | & | \\ \mathbf{a} & \mathbf{b} \\ | & | \end{bmatrix} \begin{bmatrix} \text{—} & \mathbf{c}^{\mathsf{T}} & \text{—} \\ \text{—} & \mathbf{d}^{\mathsf{T}} & \text{—} \end{bmatrix} = \mathbf{a}\mathbf{c}^{\mathsf{T}} + \mathbf{b}\mathbf{d}^{\mathsf{T}}.$$

*__5.__ Suppose A and B are symmetric. Show that AB is symmetric if and only if $AB = BA$.

6. Let A be an arbitrary $m \times n$ matrix. Show that $A^{\mathsf{T}}A$ is symmetric.

7. Explain why the matrix $A^{\mathsf{T}}A$ is a diagonal matrix whenever the column vectors of A are orthogonal to one another.

♯**8.** Suppose A is invertible. Check that $(A^{-1})^{\mathsf{T}}A^{\mathsf{T}} = I$ and $A^{\mathsf{T}}(A^{-1})^{\mathsf{T}} = I$, and deduce that A^{T} is likewise invertible with inverse $(A^{-1})^{\mathsf{T}}$.

9. If P is a permutation matrix (see Exercise 2.1.12 for the definition), show that $P^{\mathsf{T}} = P^{-1}$.

10. Suppose $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$. Check that the vector $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ satisfies $A\mathbf{y} = \mathbf{y}$ and $A^{\mathsf{T}}\mathbf{y} = \mathbf{y}$. Show that if $\mathbf{x} \cdot \mathbf{y} = 0$, then $A\mathbf{x} \cdot \mathbf{y} = 0$ as well. Interpret this result geometrically.

11. Let A be an $m \times n$ matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that if $A\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = A^{\mathsf{T}}\mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^m$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

♯**12.** Suppose A is a symmetric $n \times n$ matrix. If $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ are vectors satisfying the equations $A\mathbf{x} = 2\mathbf{x}$ and $A\mathbf{y} = 3\mathbf{y}$, show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. (*Hint:* Consider $A\mathbf{x} \cdot \mathbf{y}$.)

13. Suppose A is an $m \times n$ matrix with rank 1. Prove that there are nonzero vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ such that $A = \mathbf{u}\mathbf{v}^{\mathsf{T}}$. (*Hint:* What do the rows of $\mathbf{u}\mathbf{v}^{\mathsf{T}}$ look like?)

14. Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and its inverse matrix} \quad A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}.$$

By thinking about rows and columns of these matrices, find the inverse of

a. $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & -2 \end{bmatrix}$

♯*__15.__ Suppose A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$ satisfies $(A^{\mathsf{T}}A)\mathbf{x} = \mathbf{0}$. Prove that $A\mathbf{x} = \mathbf{0}$. (*Hint:* What is $\|A\mathbf{x}\|$?)

16. Suppose A is a symmetric matrix satisfying $A^2 = O$. Show that $A = O$. Give an example to show that the hypothesis of symmetry is required.

*__17.__ Let A_θ be the rotation matrix defined on p. 98. Using geometric reasoning, explain why $A_\theta^{-1} = A_\theta^{\mathsf{T}}$.

18. (With thanks to Maida Heatter for approximate and abbreviated recipes) To make 8 dozen *David's cookies* requires 1 lb. semisweet chocolate, 1 lb. butter, 2 c. sugar, 2 eggs, and 4 c. flour. To make 8 dozen *chocolate chip oatmeal cookies* requires 3/4 lb. semisweet chocolate, 1 lb. butter, 3 c. sugar, 2 eggs, 2 1/2 c. flour, and 6 c. oats. With the following approximate prices, what is the cost per dozen for each cookie?

Use the approach of Example 2; what are the matrices A and A^{T}?

Item	Cost
1 lb. chocolate	$4.80
1 lb. butter	3.40
1 c. sugar	0.20
1 dozen eggs	1.40
1 c. flour	0.10
1 c. oats	0.20

♯**19.** We say an $n \times n$ matrix A is *orthogonal* if $A^{\mathsf{T}}A = I_n$.

a. Prove that the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of an orthogonal matrix A are unit vectors that are orthogonal to one another, i.e.,

$$\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

b. Fill in the missing columns in the following matrices to make them orthogonal:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & ? \\ -\frac{1}{2} & ? \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & ? \\ 0 & -1 & ? \\ 0 & 0 & ? \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & ? & \frac{2}{3} \\ \frac{2}{3} & ? & -\frac{2}{3} \\ \frac{2}{3} & ? & \frac{1}{3} \end{bmatrix}$$

c. Show that any 2×2 orthogonal matrix A must be of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some real number θ. (*Hint:* Use part *a*, rather than the original definition.)

*d. Show that if A is an orthogonal 2×2 matrix, then $\mu_A : \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation or the composition of a rotation and a reflection.

e. Prove that the row vectors $\mathbf{A}_1, \dots, \mathbf{A}_n$ of an orthogonal matrix A are unit vectors that are orthogonal to one another. (*Hint:* Corollary 3.3.)

20. (Recall the definition of orthogonal matrices from Exercise 19.)

a. Show that if A and B are orthogonal $n \times n$ matrices, then so is AB.

*b. Show that if A is an orthogonal matrix, then so is A^{-1}.

21. Here is an alternative argument that when A is square and $AB = I$, it must be the case that $BA = I$ and so $B = A^{-1}$.

a. Suppose $AB = I$. Prove that A^{T} is nonsingular. (*Hint:* Solve $A^{\mathsf{T}}\mathbf{x} = \mathbf{0}$.)

b. Prove there exists a matrix C so that $A^{\mathsf{T}}C = I$, and hence $C^{\mathsf{T}}A = I$.

c. Use the result of part *c* of Exercise 2.1.11 to prove that $B = A^{-1}$.

♯**22.** a. Show that the only matrix that is both symmetric and skew-symmetric is O.

b. Given any square matrix A, show that $S = \frac{1}{2}(A + A^{\mathsf{T}})$ is symmetric and $K = \frac{1}{2}(A - A^{\mathsf{T}})$ is skew-symmetric.

c. Deduce that any square matrix A can be written in the form $A = S + K$, where S is symmetric and K is skew-symmetric.

d. Prove that the expression in part *c* is unique: If $A = S + K$ and $A = S' + K'$ (where S and S' are symmetric and K and K' are skew-symmetric), then $S = S'$ and $K = K'$. (*Hint:* Use part *a*.)

(Recall the box on p. 91.) Remember also that to prove existence (in part *c*), you need only find *some* S and K that work. There are really two different ways to prove uniqueness (in part *d*). The route suggested in the problem is to suppose there were two different solutions and show they are really the same; an alternative is to derive formulas for S and K, given the expression $A = S + K$.

23. a. Suppose A is an $m \times n$ matrix and $A\mathbf{x} \cdot \mathbf{y} = 0$ for every vector $\mathbf{x} \in \mathbb{R}^n$ and every vector $\mathbf{y} \in \mathbb{R}^m$. Prove that $A = O$.

b. Suppose A is a symmetric $n \times n$ matrix. Prove that if $A\mathbf{x} \cdot \mathbf{x} = 0$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then $A = O$. (*Hint:* Consider $A(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$.)

c. Give an example to show that the symmetry hypothesis is necessary in part *b*.

24. Suppose A is an $n \times n$ matrix satisfying $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}$ for *all* vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that A is a symmetric matrix. (*Hint:* Show that $(A - A^{\mathsf{T}})\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then use the result of part *a* of Exercise 23.)

HISTORICAL NOTES

In Chapter 1 we introduced matrices as a bookkeeping device for studying systems of linear equations, whereas in this chapter the algebra of matrices has taken on a life of its own, independent of any system of equations. Historically, the man who recognized the importance of the algebra of matrices and unified the various fragments of this theory into a subject worthy of standing by itself was Arthur Cayley (1821–1895).

Cayley was a British lawyer specializing in real estate law. He was successful but was known to say that the law was a way for him to make money so that he could pursue his true passion, mathematics. Indeed, he wrote almost 300 mathematics papers during his fourteen years of practicing law. Finally, in 1863, he sacrificed money for love and accepted a professorship at Cambridge University. Of the many hundreds of mathematics papers Cayley published during his career, the one of greatest interest here is his "*Memoir on the Theory of Matrices*," which was published in 1858 while Cayley was still practicing law. It was in this work that Cayley defined much of what you have seen in this chapter.

The term *matrix* was coined by Cayley's friend and colleague, James Joseph Sylvester (1814–1897). Many authors referred to what we now call a matrix as an "array" or "tableau." Before its mathematical definition came along, the word *matrix* was used to describe "something which surrounds, within which something is contained." A perfect word for this new object. The German mathematician F. G. Frobenius (1849–1917) had also been working with these structures, apparently without any knowledge of the work of Cayley and his colleagues. In 1878 he read Cayley's "*Memoir*" and adopted the use of the word *matrix*.

As for the notion of containment, Cayley was the first to delimit these arrays, bracketing them to emphasize that a matrix was an object to be treated as a whole. Actually, Cayley used an odd combination of curved and straight lines:

$$\begin{array}{c} (\; r \quad s \quad t \;) \\ \left| \begin{array}{ccc} u & v & w \\ x & y & z \end{array} \right| \end{array}.$$

After introducing these objects, Cayley then defined addition, subtraction, and multiplication and multiplicative inverse. Cayley's study of matrices was initially motivated by the study of linear transformations. He considered matrices as defining transformations taking quantities (x, y, z) to new quantities (X, Y, Z), and he defined matrix multiplication by composing two such transformations, just as we did in Sections 1 and 2.

It may seem that the discovery of matrices and matrix algebra was a simple bit of mathematics, but it helped lay a foundation on which a great deal of mathematics, applied mathematics, and science has been built.

CHAPTER 3

VECTOR SPACES

We return now to elaborate on the geometric discussion of solutions of systems of linear equations initiated in Chapter 1. Because every solution of a homogeneous system of linear equations is given as a linear combination of vectors, we should view the sets of solutions geometrically as generalizations of lines, planes, and hyperplanes. Intuitively, lines and planes differ in that it takes only one free variable (parameter) to describe points on a line (so a line is "one-dimensional"), but two to describe points on a plane (so a plane is "two-dimensional"). One of the goals of this chapter is to make algebraically precise the geometric notion of *dimension*, so that we may assign a dimension to every subspace of $\mathbb{R}^n$. Finally, at the end of this chapter, we shall see that these ideas extend far beyond the realm of $\mathbb{R}^n$ to the notion of an "abstract" vector space.

1 Subspaces of $\mathbb{R}^n$

In Chapter 1 we learned to write the general solution of a system of linear equations in standard form; one consequence of this procedure is that it enables us to express the solution set of a *homogeneous* system as the span of a particular set of vectors. The alert reader will realize she learned one way of reversing this process in Chapter 1, and we will learn others shortly. However, we should stop to understand that the span of a set of vectors in $\mathbb{R}^n$ and the set of solutions of a homogeneous system of linear equations share some salient properties.

Definition. A set $V \subset \mathbb{R}^n$ (a *subset* of $\mathbb{R}^n$) is called a *subspace* of $\mathbb{R}^n$ if it satisfies all the following properties:

1. $\mathbf{0} \in V$ (the zero vector belongs to V).
2. Whenever $\mathbf{v} \in V$ and $c \in \mathbb{R}$, we have $c\mathbf{v} \in V$ (V is closed under scalar multiplication).
3. Whenever $\mathbf{v}, \mathbf{w} \in V$, we have $\mathbf{v} + \mathbf{w} \in V$ (V is closed under addition).

EXAMPLE 1

Let's begin with some familiar examples.

(a) The *trivial subspace* consisting of just the zero vector $\mathbf{0} \in \mathbb{R}^n$ is a subspace, since $c\mathbf{0} = \mathbf{0}$ for any scalar c and $\mathbf{0} + \mathbf{0} = \mathbf{0}$.

(b) $\mathbb{R}^n$ itself is a subspace of $\mathbb{R}^n$.

(c) Any line ℓ through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$: If the direction vector of ℓ is $\mathbf{u} \in \mathbb{R}^n$, this means that

$$\ell = \{t\mathbf{u} : t \in \mathbb{R}\}.$$

To prove that ℓ is a subspace, we must check that the three criteria hold:

1. Setting $t = 0$, we see that $\mathbf{0} \in \ell$.
2. If $\mathbf{v} \in \ell$ and $c \in \mathbb{R}$, then $\mathbf{v} = t\mathbf{u}$ for some $t \in \mathbb{R}$, and so $c\mathbf{v} = c(t\mathbf{u}) = (ct)\mathbf{u}$, which is again a scalar multiple of $\mathbf{u}$ and hence an element of ℓ.
3. If $\mathbf{v}, \mathbf{w} \in \ell$, this means that $\mathbf{v} = s\mathbf{u}$ and $\mathbf{w} = t\mathbf{u}$ for some scalars s and t. Then $\mathbf{v} + \mathbf{w} = s\mathbf{u} + t\mathbf{u} = (s + t)\mathbf{u}$, so $\mathbf{v} + \mathbf{w} \in \ell$, as needed.

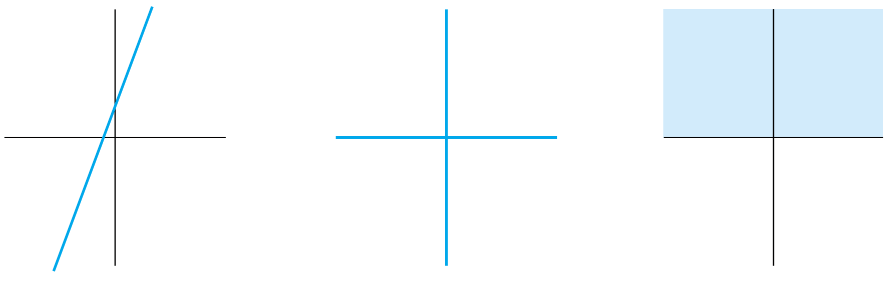
(d) Similarly, any plane through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. We leave this to the reader to check, but it is a special case of Proposition 1.2 below.

(e) Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector, and consider the hyperplane passing through the origin defined by $V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = 0\}$. Recall that $\mathbf{a}$ is the normal vector of the hyperplane. We claim that V is a subspace. As expected, we check the three criteria:

1. Since $\mathbf{a} \cdot \mathbf{0} = 0$, we conclude that $\mathbf{0} \in V$.
2. Suppose $\mathbf{v} \in V$ and $c \in \mathbb{R}$. Then $\mathbf{a} \cdot (c\mathbf{v}) = c(\mathbf{a} \cdot \mathbf{v}) = c0 = 0$, and so $c\mathbf{v} \in V$ as well.
3. Suppose $\mathbf{v}, \mathbf{w} \in V$. Then $\mathbf{a} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{a} \cdot \mathbf{v}) + (\mathbf{a} \cdot \mathbf{w}) = 0 + 0 = 0$, and therefore $\mathbf{v} + \mathbf{w} \in V$, as we needed to show. ▲

EXAMPLE 2

Let's consider next a few subsets of $\mathbb{R}^2$ that are *not* subspaces, as pictured in Figure 1.1.



(*a*) (*b*) (*c*)

Not subspaces of $\mathbb{R}^2$

FIGURE 1.1

As we commented on p. 93, to show that a multi-part definition *fails*, we only need to find *one* of the criteria that does not hold.

(a) $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 2x_1 + 1\}$ is not a subspace. All three criteria fail, but it suffices to point out $\mathbf{0} \notin S$.

(b) $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$ is not a subspace. Each of the vectors $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (0, 1)$ lies in S, and yet their sum $\mathbf{v} + \mathbf{w} = (1, 1)$ does not.

(c) $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ is not a subspace. The vector $\mathbf{v} = (0, 1)$ lies in S, and yet any negative scalar multiple of it, e.g., $(-2)\mathbf{v} = (0, -2)$, does not. ▲

We now return to our motivating discussion. First, we consider the solution set of a homogeneous linear system.

Proposition 1.1. *Let A be an $m \times n$ matrix, and consider the set of solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$; that is, let*

$$V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Then V is a subspace of $\mathbb{R}^n$.

Proof. The proof is essentially the same as Example 1(e) if we think of the equation $A\mathbf{x} = \mathbf{0}$ as being the collection of equations $A_1 \cdot \mathbf{x} = A_2 \cdot \mathbf{x} = \cdots = A_m \cdot \mathbf{x} = 0$. But we would rather phrase the argument in terms of the linearity properties of matrix multiplication, discussed in Section 1 of Chapter 2.

As usual, we need only check that the three defining criteria all hold.

1. To check that $\mathbf{0} \in V$, we recall that $A\mathbf{0} = \mathbf{0}$, as a consequence of either of our ways of thinking of matrix multiplication.
2. If $\mathbf{v} \in V$ and $c \in \mathbb{R}$, then we must show that $c\mathbf{v} \in V$. Well, $A(c\mathbf{v}) = c(A\mathbf{v}) = c\mathbf{0} = \mathbf{0}$.
3. If $\mathbf{v}, \mathbf{w} \in V$, then we must show that $\mathbf{v} + \mathbf{w} \in V$. Since $A\mathbf{v} = A\mathbf{w} = \mathbf{0}$, we have $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, as required.

Thus, V is indeed a subspace of $\mathbb{R}^n$. □

Next, let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. In Chapter 1 we defined $\text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$ to be the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$; that is,

$$\text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \text{ for some scalars } c_1, \dots, c_k\}.$$

Generalizing what we observed in Examples 1(c) and (d), we have the following proposition.

Proposition 1.2. *Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. Then $V = \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.*

Proof. We check that all three criteria hold.

1. To see that $\mathbf{0} \in V$, we merely take $c_1 = c_2 = \cdots = c_k = 0$. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$.
2. Suppose $\mathbf{v} \in V$ and $c \in \mathbb{R}$. By definition, there are scalars $c_1, \dots, c_k$ so that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. Thus,
$$c\mathbf{v} = c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdots + (cc_k)\mathbf{v}_k,$$
which is again a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, so $c\mathbf{v} \in V$, as desired.
3. Suppose $\mathbf{v}, \mathbf{w} \in V$. This means there are scalars $c_1, \dots, c_k$ and $d_1, \dots, d_k$ so that[1]
$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \quad \text{and} \quad \mathbf{w} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k;$$

[1] This might be a good time to review the content of the box following Exercise 1.1.22.

adding, we obtain

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= (c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k)+(d_1\mathbf{v}_1+\cdots+d_k\mathbf{v}_k)\\ &= (c_1+d_1)\mathbf{v}_1+\cdots+(c_k+d_k)\mathbf{v}_k,\end{aligned}$$

which is again a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ and hence an element of V.

This completes the verification that V is a subspace of $\mathbb{R}^n$. □

Remark. Let $V \subset \mathbb{R}^n$ be a subspace and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$. Then of course the subspace $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is a subset of V. We say that $\mathbf{v}_1, \dots, \mathbf{v}_k$ *span* V if $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = V$. (The point here is that *every* vector in V must be a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.)

EXAMPLE 3

The plane

$$\mathcal{P}_1 = \left\{ s\begin{bmatrix}1\\-1\\2\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix} : s, t \in \mathbb{R}\right\}$$

is the span of the vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \quad\text{and}\quad \mathbf{v}_2 = \begin{bmatrix}2\\0\\1\end{bmatrix}$$

and is therefore a subspace of $\mathbb{R}^3$. On the other hand, the plane

$$\mathcal{P}_2 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\-1\\2\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix} : s, t \in \mathbb{R}\right\}$$

is not a subspace. This is most easily verified by checking that $\mathbf{0} \notin \mathcal{P}_2$. Well, $\mathbf{0} \in \mathcal{P}_2$ precisely when we can find values of s and t such that

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\-1\\2\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix}.$$

This amounts to the system of equations

$$\begin{aligned} s + 2t &= -1\\ -s &= 0\\ 2s + t &= 0,\end{aligned}$$

which we easily see is inconsistent.

A word of warning here: We might have expressed $\mathcal{P}_1$ in the form

$$\left\{ \begin{bmatrix}1\\1\\-1\end{bmatrix} + s\begin{bmatrix}1\\-1\\2\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix} : s, t \in \mathbb{R}\right\};$$

the presence of the "shifting" term may not prevent the plane from passing through the origin. ▲

EXAMPLE 4

Let

$$\mathcal{P}_1 = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right) \quad \text{and} \quad \mathcal{P}_2 = \text{Span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right).$$

We wish to find all the vectors contained in *both* $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$.

A vector $\mathbf{x}$ lies in both $\mathcal{P}_1$ and $\mathcal{P}_2$ if and only if we can write $\mathbf{x}$ in both the forms

$$\mathbf{x} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} \quad \text{and} \quad \mathbf{x} = c\begin{bmatrix}-1\\1\\0\end{bmatrix} + d\begin{bmatrix}2\\1\\2\end{bmatrix}$$

for some scalars a, b, c, and d. Setting the two expressions for $\mathbf{x}$ equal to one another and moving all the vectors to one side, we obtain the system of equations

$$-a\begin{bmatrix}1\\0\\0\end{bmatrix} - b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\\0\end{bmatrix} + d\begin{bmatrix}2\\1\\2\end{bmatrix} = \mathbf{0}.$$

In other words, we want to find all solutions of the system

$$\begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & 1\\0 & 1 & 0 & 2\end{bmatrix}\begin{bmatrix}-a\\-b\\c\\d\end{bmatrix} = \mathbf{0},$$

and so we reduce the matrix

$$A = \begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & 1\\0 & 1 & 0 & 2\end{bmatrix}$$

to reduced echelon form

$$R = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{bmatrix}$$

and find that every solution of $A\mathbf{y} = 0$ is a scalar multiple of the vector

$$\begin{bmatrix}-a\\-b\\c\\d\end{bmatrix} = \begin{bmatrix}-1\\-2\\1\\1\end{bmatrix}.$$

This means that

$$\mathbf{x} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\1\end{bmatrix} = 1\begin{bmatrix}-1\\1\\0\end{bmatrix} + 1\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

spans the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$. We expected such a result on geometric grounds, since the intersection of two distinct planes through the origin in $\mathbb{R}^3$ should be a line. ▲

We ask the reader to show in Exercise 6 that, more generally, the intersection of subspaces is again always a subspace. We now investigate some other ways to concoct new subspaces from old.

EXAMPLE 5

Let U and V be subspaces of $\mathbb{R}^n$. We define their *sum* to be

$$U + V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}.$$

That is, $U + V$ consists of all vectors that can be obtained by adding *some* vector in U to *some* vector in V, as shown in Figure 1.2. Be careful to note that, unless one of U or V is

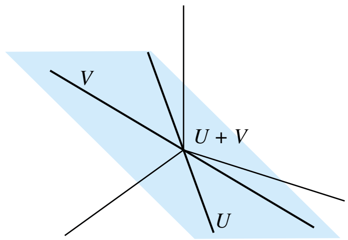


FIGURE 1.2

contained in the other, $U + V$ is much larger than $U \cup V$. We check that if U and V are subspaces, then $U + V$ is again a subspace:

1. Since $\mathbf{0} \in U$ and $\mathbf{0} \in V$, we have $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + V$.
2. Suppose $\mathbf{x} \in U + V$ and $c \in \mathbb{R}$. We are to show that $c\mathbf{x} \in U + V$. By definition, $\mathbf{x}$ can be written in the form
$$\mathbf{x} = \mathbf{u} + \mathbf{v} \qquad \text{for some } \mathbf{u} \in U \quad \text{and} \quad \mathbf{v} \in V.$$
Then we have
$$c\mathbf{x} = c(\mathbf{u} + \mathbf{v}) = (c\mathbf{u}) + (c\mathbf{v}) \in U + V,$$
noting that $c\mathbf{u} \in U$ and $c\mathbf{v} \in V$ since each of U and V is closed under scalar multiplication.
3. Suppose $\mathbf{x}, \mathbf{y} \in U + V$. Then
$$\mathbf{x} = \mathbf{u} + \mathbf{v} \quad \text{and} \quad \mathbf{y} = \mathbf{u}' + \mathbf{v}' \qquad \text{for some } \mathbf{u}, \mathbf{u}' \in U \quad \text{and} \quad \mathbf{v}, \mathbf{v}' \in V.$$
Therefore, we have
$$\mathbf{x} + \mathbf{y} = (\mathbf{u} + \mathbf{v}) + (\mathbf{u}' + \mathbf{v}') = (\mathbf{u} + \mathbf{u}') + (\mathbf{v} + \mathbf{v}') \in U + V,$$
noting that $\mathbf{u} + \mathbf{u}' \in U$ and $\mathbf{v} + \mathbf{v}' \in V$ since U and V are both closed under addition.

Thus, as required, $U + V$ is a subspace. Indeed, it is the smallest subspace containing both U and V. (See Exercise 7.) ▲

Given an $m \times n$ matrix A, we can think of the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ as the set of all vectors that are orthogonal to each of the row vectors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m$, and, hence, by Exercise 1.2.11, are orthogonal to every vector in $V = \text{Span}\,(\mathbf{A}_1, \dots, \mathbf{A}_m)$. This leads us to a very important and natural notion.

Definition. Given a subspace $V \subset \mathbb{R}^n$, define

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = 0 \ \text{ for every } \mathbf{v} \in V\}.$$

$V^\perp$ (read "V perp") is called the *orthogonal complement* of V.[2] (See Figure 1.3.)

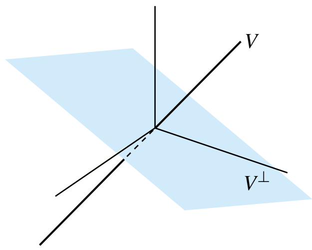


FIGURE 1.3

Proposition 1.3. *$V^\perp$ is a subspace of $\mathbb{R}^n$.*

Proof. We check the requisite three properties.

1. $\mathbf{0} \in V^\perp$ because $\mathbf{0} \cdot \mathbf{v} = 0$ for every $\mathbf{v} \in V$.
2. Suppose $\mathbf{x} \in V^\perp$ and $c \in \mathbb{R}$. We must check that $c\mathbf{x} \in V^\perp$. We calculate
$$(c\mathbf{x}) \cdot \mathbf{v} = c(\mathbf{x} \cdot \mathbf{v}) = 0$$
for all $\mathbf{v} \in V$, as required.
3. Suppose $\mathbf{x}, \mathbf{y} \in V^\perp$; we must check that $\mathbf{x} + \mathbf{y} \in V^\perp$. Well,
$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = (\mathbf{x} \cdot \mathbf{v}) + (\mathbf{y} \cdot \mathbf{v}) = 0 + 0 = 0$$
for all $\mathbf{v} \in V$, as needed. □

EXAMPLE 6

Let $V = \text{Span}\big((1, 2, 1)\big) \subset \mathbb{R}^3$. Then $V^\perp$ is by definition the plane $W = \{\mathbf{x} : x_1 + 2x_2 + x_3 = 0\}$. And what is $W^\perp$? Clearly, any multiple of $(1, 2, 1)$ must be orthogonal to every vector in W; but is $\text{Span}\big((1, 2, 1)\big)$ *all* of $W^\perp$? Common sense suggests that the answer is yes, but let's be sure.

We know that the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

span W (why?), so we can find $W^\perp$ by solving the equations

$$(-2, 1, 0) \cdot \mathbf{x} = (-1, 0, 1) \cdot \mathbf{x} = 0.$$

[2] In fact, both this definition and Proposition 1.3 work just fine for any sub<u>set</u> $V \subset \mathbb{R}^n$.

By finding the reduced echelon form of the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix},$$

we see that, indeed, every vector in $W^\perp$ is a multiple of $(1, 2, 1)$, as we suspected. ▲

It is extremely important to observe that if $\mathbf{c} \in V^\perp$, then all the elements of V satisfy the linear equation $\mathbf{c} \cdot \mathbf{x} = 0$. Thus, there is an intimate relation between elements of $V^\perp$ and Cartesian equations defining the subspace V. We will explore and exploit this relation more fully in the next few sections.

It will be useful for us to make the following definition.

> ***Definition.*** Let V and W be subspaces of $\mathbb{R}^n$. We say V and W are *orthogonal subspaces* if every element of V is orthogonal to every element of W, i.e., if
>
> $$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for every } \mathbf{v} \in V \text{ and every } \mathbf{w} \in W.$$

Remark. If $V = W^\perp$ or $W = V^\perp$, then clearly V and W are orthogonal subspaces. On the other hand, if V and W are orthogonal subspaces of $\mathbb{R}^n$, then certainly $W \subset V^\perp$ and $V \subset W^\perp$. (See Exercise 12.) Of course, W need not be equal to $V^\perp$: Consider, for example, V to be the x_1-axis and W to be the x_2-axis in $\mathbb{R}^3$. Then $V^\perp$ is the x_2x_3-plane, which contains W and more. It is natural, however, to ask the following question: If $W = V^\perp$, must $V = W^\perp$? We will return to this shortly.

Exercises 3.1

*__1.__ Which of the following are subspaces? Justify your answer in each case.

a. $\{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}$

b. $\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \text{ for some } a, b \in \mathbb{R}\right\}$

c. $\{\mathbf{x} \in \mathbb{R}^3 : x_1 + 2x_2 < 0\}$

d. $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$

e. $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 0\}$

f. $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = -1\}$

g. $\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R}\right\}$

h. $\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R}\right\}$

i. $\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R}\right\}$

*2. Decide whether each of the following collections of vectors spans $\mathbb{R}^3$.
   a. $(1, 1, 1), (1, 2, 2)$
   b. $(1, 1, 1), (1, 2, 2), (1, 3, 3)$
   c. $(1, 0, 1), (1, -1, 1), (3, 5, 3), (2, 3, 2)$
   d. $(1, 0, -1), (2, 1, 1), (0, 1, 5)$

*3. Criticize the following argument: For any vector $\mathbf{v}$, we have $0\mathbf{v} = \mathbf{0}$. So the first criterion for subspaces is, in fact, a consequence of the second criterion and could therefore be omitted.

4. Let A be an $n \times n$ matrix. Verify that
$$V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 3\mathbf{x}\}$$
is a subspace of $\mathbb{R}^n$.

5. Let A and B be $m \times n$ matrices. Show that
$$V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = B\mathbf{x}\}$$
is a subspace of $\mathbb{R}^n$.

6. a. Let U and V be subspaces of $\mathbb{R}^n$. Define the *intersection* of U and V to be
$$U \cap V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ and } \mathbf{x} \in V\}.$$
   Show that $U \cap V$ is a subspace of $\mathbb{R}^n$. Give two examples.
   b. Is $U \cup V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ or } \mathbf{x} \in V\}$ always a subspace of $\mathbb{R}^n$? Give a proof or counterexample.

7. Prove that if U and V are subspaces of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ containing all the vectors of U and all the vectors of V (that is, $U \subset W$ and $V \subset W$), then $U + V \subset W$. This means that $U + V$ is the smallest subspace containing both U and V.

8. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and let $\mathbf{v} \in \mathbb{R}^n$. Prove that
$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}) \quad \text{if and only if} \quad \mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

9. Determine the intersection of the subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$ in each case:
   *a. $\mathcal{P}_1 = \text{Span}\big((1, 0, 1), (2, 1, 2)\big)$, $\mathcal{P}_2 = \text{Span}\big((1, -1, 0), (1, 3, 2)\big)$
   b. $\mathcal{P}_1 = \text{Span}\big((1, 2, 2), (0, 1, 1)\big)$, $\mathcal{P}_2 = \text{Span}\big((2, 1, 1), (1, 0, 0)\big)$
   c. $\mathcal{P}_1 = \text{Span}\big((1, 0, -1), (1, 2, 3)\big)$, $\mathcal{P}_2 = \{\mathbf{x} : x_1 - x_2 + x_3 = 0\}$
   *d. $\mathcal{P}_1 = \text{Span}\big((1, 1, 0, 1), (0, 1, 1, 0)\big)$, $\mathcal{P}_2 = \text{Span}\big((0, 0, 1, 1), (1, 1, 0, 0)\big)$
   e. $\mathcal{P}_1 = \text{Span}\big((1, 0, 1, 2), (0, 1, 0, -1)\big)$, $\mathcal{P}_2 = \text{Span}\big((1, 1, 2, 1), (1, 1, 0, 1)\big)$

♯*10. Let $V \subset \mathbb{R}^n$ be a subspace. Show that $V \cap V^\perp = \{\mathbf{0}\}$.

11. Suppose V and W are orthogonal subspaces of $\mathbb{R}^n$, i.e., $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and every $\mathbf{w} \in W$. Prove that $V \cap W = \{\mathbf{0}\}$.

♯*12. Suppose V and W are orthogonal subspaces of $\mathbb{R}^n$, i.e., $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and every $\mathbf{w} \in W$. Prove that $V \subset W^\perp$.

♯13. Let $V \subset \mathbb{R}^n$ be a subspace. Show that $V \subset (V^\perp)^\perp$. Do you think more is true?

♯14. Let V and W be subspaces of $\mathbb{R}^n$ with the property that $V \subset W$. Prove that $W^\perp \subset V^\perp$.

15. Let A be an $m \times n$ matrix. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be subspaces.
   a. Show that $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \in W\}$ is a subspace of $\mathbb{R}^n$.
   b. Show that $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in V\}$ is a subspace of $\mathbb{R}^m$.

16. Suppose A is a symmetric $n \times n$ matrix. Let $V \subset \mathbb{R}^n$ be a subspace with the property that $A\mathbf{x} \in V$ for every $\mathbf{x} \in V$. Show that $A\mathbf{y} \in V^\perp$ for all $\mathbf{y} \in V^\perp$.

17. Use Exercises 13 and 14 to prove that for any subspace $V \subset \mathbb{R}^n$, we have $V^\perp = \big((V^\perp)^\perp\big)^\perp$.

18. Suppose U and V are subspaces of $\mathbb{R}^n$. Prove that $(U + V)^\perp = U^\perp \cap V^\perp$.

2 The Four Fundamental Subspaces

As we have seen, two of the most important constructions we've studied in linear algebra—the span of a collection of vectors and the set of solutions of a homogeneous linear system of equations—lead to subspaces. Let's use these notions to define four important subspaces associated to an $m \times n$ matrix.

The first two are already quite familiar to us from our work in Chapter 1, and we have seen in Section 1 of this chapter that they are in fact subspaces. Here we will give them their official names.

Definition (Nullspace). Let A be an $m \times n$ matrix. The *nullspace* of A is the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Definition (Column Space). Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$. We define the *column space* of A to be the subspace of $\mathbb{R}^m$ spanned by the column vectors:

$$\mathbf{C}(A) = \mathrm{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_n) \subset \mathbb{R}^m.$$

Of course, the nullspace, $\mathbf{N}(A)$, is just the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ that we first encountered in Section 4 of Chapter 1. What is less obvious is that we encountered the column space, $\mathbf{C}(A)$, in Section 5 of Chapter 1, as we now see.

Proposition 2.1. *Let A be an $m \times n$ matrix. Let $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. That is,*

$$\mathbf{C}(A) = \{\mathbf{b} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ is consistent}\}.$$

Proof. By definition, $\mathbf{C}(A) = \mathrm{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_n)$, and so $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$; i.e., $\mathbf{b} = x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n$ for some scalars $x_1, \dots, x_n$. Recalling our crucial observation ($*$) on p. 53, we conclude that $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. The final reformulation is straightforward so long as we remember that the system $A\mathbf{x} = \mathbf{b}$ is consistent provided it has a solution. □

Remark. If, as in Section 1 of Chapter 2, we think of A as giving a function $\mu_A : \mathbb{R}^n \to \mathbb{R}^m$, then $\mathbf{C}(A) \subset \mathbb{R}^m$ is the set of all the values of the function μ_A, i.e., the *image* of μ_A. It is important to keep track of where each subspace "lives" as you continue through this chapter: The nullspace $\mathbf{N}(A)$ consists of $\mathbf{x}$'s (inputs of μ_A) and is a subspace of $\mathbb{R}^n$; the column space $\mathbf{C}(A)$ consists of $\mathbf{b}$'s (outputs of the function μ_A) and is a subspace of $\mathbb{R}^m$.

A theme we explored in Chapter 1 was that lines and planes can be described either parametrically or by Cartesian equations. This idea should work for general subspaces of $\mathbb{R}^n$. We give a *parametric* description of a subspace V when we describe V as the span of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. Putting these vectors as the columns of a matrix A amounts to writing $V = \mathbf{C}(A)$. Similarly, giving *Cartesian equations* for V, once we translate them into matrix form, is giving $V = \mathbf{N}(A)$ for the appropriate matrix A.[3] Much of Sections 4 and 5 of Chapter 1 was devoted to going from one description to the other: In our present language, by finding the general solution of $A\mathbf{x} = \mathbf{0}$, we obtain a parametric description of $\mathbf{N}(A)$ and thus obtain vectors that span that subspace. On the other hand, finding the constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent provides a set of Cartesian equations for $\mathbf{C}(A)$.

EXAMPLE 1

Let

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix}.$$

Of course, we bring A to its reduced echelon form

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and read off the general solution of $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= \phantom{-x_3 - {}} x_4 \\ x_3 &= x_3 \\ x_4 &= \phantom{-x_3 - {}} x_4\,, \end{aligned}$$

that is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

From this we see that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbf{N}(A)$.

On other hand, we know that the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ span $\mathbf{C}(A)$. To find Cartesian equations for $\mathbf{C}(A)$, we find the constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent

[3] The astute reader may be worried that we have not yet shown that *every* subspace can be described in either manner. We will address this matter in Section 4.

by reducing the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & 1 & b_2 \\ 1 & 2 & 1 & -1 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & b_1 \\ 0 & 1 & 0 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 2b_1 - 3b_2 + b_3 \end{array}\right],$$

from which we see that $2b_1 - 3b_2 + b_3 = 0$ gives a Cartesian description of $\mathbf{C}(A)$. Of course, we might want to replace b's with x's and just write

$$\mathbf{C}(A) = \{\mathbf{x} \in \mathbb{R}^3 : 2x_1 - 3x_2 + x_3 = 0\}.$$

We can summarize these results by defining new matrices

$$X = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix},$$

and then we have $\mathbf{N}(A) = \mathbf{C}(X)$ and $\mathbf{C}(A) = \mathbf{N}(Y)$. One final remark: Note that the coefficients of the constraint equation(s), i.e., the row(s) of Y, give vectors orthogonal to $\mathbf{C}(A)$, just as the rows of A are orthogonal to $\mathbf{N}(A)$ (and hence to the columns of X). ▲

We now move on to discuss the last two of the four subspaces associated to the matrix A. In the interest of fair play, since we've already dedicated a subspace to the columns of A, it is natural to make the following definition.

Definition (Row Space). Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{R}^n$. We define the *row space* of A to be the subspace of $\mathbb{R}^n$ spanned by the row vectors $\mathbf{A}_1, \dots, \mathbf{A}_m$:

$$\mathbf{R}(A) = \text{Span}\,(\mathbf{A}_1, \dots, \mathbf{A}_m) \subset \mathbb{R}^n.$$

It is important to remember that, as vectors in $\mathbb{R}^n$, the $\mathbf{A}_i$ are still represented by column vectors with n entries. But we continue our practice of writing vectors in parentheses when it is typographically more convenient.

Noting that $\mathbf{R}(A) = \mathbf{C}(A^{\mathsf{T}})$, it is natural then to complete the quartet as follows:

Definition (Left Nullspace). We define the *left nullspace* of the $m \times n$ matrix A to be

$$\mathbf{N}(A^{\mathsf{T}}) = \{\mathbf{x} \in \mathbb{R}^m : A^{\mathsf{T}}\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x}^{\mathsf{T}} A = \mathbf{0}^{\mathsf{T}}\}.$$

(The latter description accounts for the terminology.)

Just as elements of the nullspace of A give us the linear combinations of the *column* vectors of A that result in the zero vector, elements of the left nullspace give us the linear combinations of the *row* vectors of A that result in zero.

Once again, we pause to remark on the "locations" of the subspaces. $\mathbf{N}(A)$ and $\mathbf{R}(A)$ are "neighbors," both being subspaces of $\mathbb{R}^n$ (the domain of the linear map μ_A). $\mathbf{C}(A)$ and $\mathbf{N}(A^{\mathsf{T}})$ are "neighbors" in $\mathbb{R}^m$, the range of μ_A and the domain of $\mu_{A^{\mathsf{T}}}$. We will soon have a more complete picture of the situation.

In the discussion leading up to Proposition 1.3 we observed that vectors in the nullspace of A are orthogonal to all the row vectors of A—that is, that $\mathbf{N}(A)$ and $\mathbf{R}(A)$ are orthogonal subspaces. In fact, the orthogonality relations among our "neighboring" subspaces will provide a lot of information about linear maps. We begin with the following proposition.

Proposition 2.2. *Let A be an $m \times n$ matrix. Then* $\mathbf{N}(A) = \mathbf{R}(A)^\perp$.

Proof. If $\mathbf{x} \in \mathbf{N}(A)$, then $\mathbf{x}$ is orthogonal to each row vector $\mathbf{A}_1, \dots, \mathbf{A}_m$ of A. By Exercise 1.2.11, $\mathbf{x}$ is orthogonal to every vector in $\mathbf{R}(A)$ and is therefore an element of $\mathbf{R}(A)^\perp$. Thus, $\mathbf{N}(A)$ is a subset of $\mathbf{R}(A)^\perp$, and so we need only show that $\mathbf{R}(A)^\perp$ is a subset of $\mathbf{N}(A)$. (Recall the box on p. 12.) If $\mathbf{x} \in \mathbf{R}(A)^\perp$, this means that $\mathbf{x}$ is orthogonal to every vector in $\mathbf{R}(A)$, so, in particular, $\mathbf{x}$ is orthogonal to each of the row vectors $\mathbf{A}_1, \dots, \mathbf{A}_m$. But this means that $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \mathbf{N}(A)$, as required. □

Since $\mathbf{C}(A) = \mathbf{R}(A^\top)$, when we substitute $A^\top$ for A the following result is an immediate consequence of Proposition 2.2.

Proposition 2.3. *Let A be an $m \times n$ matrix. Then* $\mathbf{N}(A^\top) = \mathbf{C}(A)^\perp$.

Proposition 2.3 has a very pleasant interpretation in terms of the constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent—the Cartesian equations for $\mathbf{C}(A)$. As we commented in Section 1, the coefficients of such a Cartesian equation give a vector orthogonal to $\mathbf{C}(A)$, i.e., an element of $\mathbf{C}(A)^\perp = \mathbf{N}(A^\top)$. Thus, a constraint equation gives a linear combination of the rows that results in the zero vector. But, of course, this is where constraint equations come from in the first place. Conversely, any such relation among the row vectors of A gives an element of $\mathbf{N}(A^\top) = \mathbf{C}(A)^\perp$, and hence the coefficients of a constraint equation that $\mathbf{b}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to be consistent.

EXAMPLE 2

Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

We find the constraint equations for $A\mathbf{x} = \mathbf{b}$ to be consistent by row reducing the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 1 & 1 & b_2 \\ 0 & 1 & b_3 \\ 1 & 2 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & b_2 - b_1 \\ 0 & 1 & b_3 \\ 0 & 0 & b_4 - b_1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - b_2 \\ 0 & 0 & -b_1 + b_2 + b_3 \\ 0 & 0 & -b_1 + b_4 \end{array}\right].$$

The constraint equations are

$$\begin{aligned} -b_1 + b_2 + b_3 &= 0 \\ -b_1 + b_4 &= 0. \end{aligned}$$

Note that the vectors

$$\mathbf{c}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are in $\mathbf{N}(A^\top)$ and correspond to linear combinations of the rows yielding $\mathbf{0}$. ▲

Proposition 2.3 tells us that $\mathbf{N}(A^{\top}) = \mathbf{C}(A)^{\perp}$, and so $\mathbf{N}(A^{\top})$ and $\mathbf{C}(A)$ are orthogonal subspaces. It is natural, then, to ask whether $\mathbf{N}(A^{\top})^{\perp} = \mathbf{C}(A)$, as well.

Proposition 2.4. *Let A be an $m \times n$ matrix. Then $\mathbf{C}(A) = \mathbf{N}(A^{\top})^{\perp}$.*

Proof. Since $\mathbf{C}(A)$ and $\mathbf{N}(A^{\top})$ are orthogonal subspaces, we infer from Exercise 3.1.12 that $\mathbf{C}(A) \subset \mathbf{N}(A^{\top})^{\perp}$. On the other hand, from Section 5 of Chapter 1 we know that there is a system of constraint equations

$$\mathbf{c}_1 \cdot \mathbf{b} = \cdots = \mathbf{c}_k \cdot \mathbf{b} = 0$$

that give necessary and sufficient conditions for $\mathbf{b} \in \mathbb{R}^m$ to belong to $\mathbf{C}(A)$. Setting $V = \text{Span}\,(\mathbf{c}_1, \ldots, \mathbf{c}_k) \subset \mathbb{R}^m$, this means that $\mathbf{C}(A) = V^{\perp}$. Since each such vector $\mathbf{c}_j$ is an element of $\mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\top})$, we conclude that $V \subset \mathbf{N}(A^{\top})$. It follows from Exercise 3.1.14 that $\mathbf{N}(A^{\top})^{\perp} \subset V^{\perp} = \mathbf{C}(A)$. Combining the two inclusions, we have $\mathbf{C}(A) = \mathbf{N}(A^{\top})^{\perp}$, as required. □

Now that we have proved Proposition 2.4, we can complete the circle of ideas. We have the following result, summarizing the geometric relations of the pairs of the four fundamental subspaces.

Theorem 2.5. *Let A be an $m \times n$ matrix. Then*

1. $\mathbf{R}(A)^{\perp} = \mathbf{N}(A)$
2. $\mathbf{N}(A)^{\perp} = \mathbf{R}(A)$
3. $\mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\top})$
4. $\mathbf{N}(A^{\top})^{\perp} = \mathbf{C}(A)$

Proof. All but the second are the contents of Propositions 2.2, 2.3, and 2.4. The second follows from Proposition 2.4 by substituting $A^{\top}$ for A. □

Figure 2.1 is a schematic diagram giving a visual representation of these results.

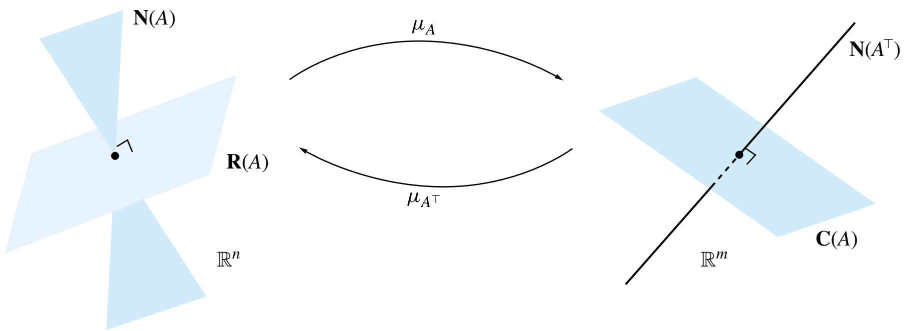


FIGURE 2.1

Remark. Combining these pairs of results, we conclude that for any of the four fundamental subspaces $V = \mathbf{R}(A), \mathbf{N}(A), \mathbf{C}(A)$, and $\mathbf{N}(A^{\top})$, it is the case that $(V^{\perp})^{\perp} = V$. If we knew that every subspace of $\mathbb{R}^n$ could be so written, we would have the result in general; this will come soon.

EXAMPLE 3

Let's look for matrices whose row spaces are the plane in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and satisfy the extra conditions given below. Note, first of all, that these must be $m \times 3$ matrices for some positive integers m.

(a) Suppose we want such a matrix A with $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ in its nullspace. Remember that $\mathbf{N}(A) = \mathbf{R}(A)^\perp$. We cannot succeed: Although $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, it is not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and hence not orthogonal to every vector in the row space.

(b) Suppose we want such a matrix with its column space equal to $\mathbb{R}^2$. Now we win: We need a 2×3 matrix, and we just try

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Then $\mathbf{C}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \mathbb{R}^2$, as required.

(c) Suppose we want such a matrix A whose column space is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This seems impossible, but here's an argument to that effect. If we had such a matrix A, note that $\mathbf{C}(A)^\perp = \mathbf{N}(A^\mathsf{T})$ is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so we would have to have $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{0}$. This means that the row space of A is a line.

(d) Following this reasoning, let's look for a matrix A whose column space is spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. We note that A now must be a 3×3 matrix. As before, note that $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in \mathbf{C}(A)^\perp = \mathbf{N}(A^\mathsf{T})$, and so the third row of A must be the sum of the first two rows. So now we just try

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Perhaps it's not obvious that A really works, but if we add the first and second columns, we get $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, and if we subtract them we get $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, so $\mathbf{C}(A)$ contains both the desired vectors and hence their span. We leave it to the reader to check that $\mathbf{C}(A)$ is not larger than this span. ▲

Exercises 3.2

♯*1. Show that if B is obtained from A by performing one or more row operations, then $\mathbf{R}(B) = \mathbf{R}(A)$.

2. What vectors $\mathbf{b}$ are in the column space of A in each case? (Give constraint equations.) Check that the coefficients of the constraint equations give linear combinations of the rows of A summing to $\mathbf{0}$.

*a. $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix}$ *b. $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & -5 \end{bmatrix}$ c. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ -1 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

3. Given each matrix A, find matrices X and Y so that $\mathbf{C}(A) = \mathbf{N}(X)$ and $\mathbf{N}(A) = \mathbf{C}(Y)$.

*a. $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

*4. Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 0 & 3 & 4 \\ 2 & 2 & -2 & -3 \end{bmatrix}$.

a. Give constraint equations for $\mathbf{C}(A)$.

b. Find vectors spanning $\mathbf{N}(A^{\mathsf{T}})$.

*5. Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $A = LU$, give vectors that span $\mathbf{R}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A)$.

6. a. Construct a matrix whose column space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and whose nullspace contains $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, or explain why none can exist.

*b. Construct a matrix whose column space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and whose nullspace contains $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, or explain why none can exist.

7. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$.

a. Give $\mathbf{C}(A)$ and $\mathbf{C}(B)$. Are they lines, planes, or all of $\mathbb{R}^3$?

b. Describe $\mathbf{C}(A+B)$ and $\mathbf{C}(A)+\mathbf{C}(B)$. Compare your answers.

8. *a. Construct a 3×3 matrix A with $\mathbf{C}(A) \subset \mathbf{N}(A)$.

b. Construct a 3×3 matrix A with $\mathbf{N}(A) \subset \mathbf{C}(A)$.

c. Do you think there can be a 3×3 matrix A with $\mathbf{N}(A) = \mathbf{C}(A)$? Why or why not?

d. Construct a 4×4 matrix A with $\mathbf{C}(A) = \mathbf{N}(A)$.

***9.** Let A be an $m \times n$ matrix and recall that we have the associated function $\mu_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mu_A(\mathbf{x}) = A\mathbf{x}$. Show that μ_A is a one-to-one function if and only if $\mathbf{N}(A) = \{\mathbf{0}\}$.

♯**10.** Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

a. $\mathbf{N}(B) \subset \mathbf{N}(AB)$.

b. $\mathbf{C}(AB) \subset \mathbf{C}(A)$. (*Hint:* Use Proposition 2.1.)

c. $\mathbf{N}(B) = \mathbf{N}(AB)$ when A is $n \times n$ and nonsingular. (*Hint:* See the box on p. 12.)

d. $\mathbf{C}(AB) = \mathbf{C}(A)$ when B is $n \times n$ and nonsingular.

♯**11.** Let A be an $m \times n$ matrix. Prove that $\mathbf{N}(A^{\mathsf{T}}A) = \mathbf{N}(A)$. (*Hint:* Use Exercise 10 and Exercise 2.5.15.)

12. Suppose A and B are $m \times n$ matrices. Prove that $\mathbf{C}(A)$ and $\mathbf{C}(B)$ are orthogonal subspaces of $\mathbb{R}^m$ if and only if $A^{\mathsf{T}}B = O$.

13. Suppose A is an $n \times n$ matrix with the property that $A^2 = A$.

a. Prove that $\mathbf{C}(A) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = A\mathbf{x}\}$.

b. Prove that $\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{u} - A\mathbf{u} \text{ for some } \mathbf{u} \in \mathbb{R}^n\}$.

c. Prove that $\mathbf{C}(A) \cap \mathbf{N}(A) = \{\mathbf{0}\}$.

d. Prove that $\mathbf{C}(A) + \mathbf{N}(A) = \mathbb{R}^n$.

3 Linear Independence and Basis

In view of our discussion in the preceding section, it is natural to ask the following question:

Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, is $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$?

Of course, we recognize that this is a question of whether there *exist* scalars $c_1, \dots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. As we are well aware, this is, in turn, a question of whether a certain (inhomogeneous) system of linear equations has a solution. As we saw in Chapter 1, one is often interested in the allied question: Is that solution *unique*?

EXAMPLE 1

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We ask first of all whether $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. This is a familiar question when we recast it in matrix notation: Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Is the system $A\mathbf{x} = \mathbf{b}$ consistent? Immediately we write down the appropriate augmented matrix and reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right],$$

so the system is obviously inconsistent. The answer is: No, $\mathbf{v}$ is not in $\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

What about

$$\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}?$$

As the reader can easily check, $\mathbf{w} = 3\mathbf{v}_1 - \mathbf{v}_3$, so $\mathbf{w} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. What's more, $\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$, as well. So, obviously, there is no unique expression for $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. But we can conclude more: Setting the two expressions for $\mathbf{w}$ equal, we obtain

$$3\mathbf{v}_1 - \mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3, \quad \text{i.e.,} \quad \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 = \mathbf{0}.$$

That is, there is a nontrivial relation among the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and this is why we have different ways of expressing $\mathbf{w}$ as a linear combination of the three of them. Indeed, because $\mathbf{v}_1 = -\mathbf{v}_2 + 2\mathbf{v}_3$, we can see easily that any linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is a linear combination of just $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1(-\mathbf{v}_2 + 2\mathbf{v}_3) + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_2 - c_1)\mathbf{v}_2 + (c_3 + 2c_1)\mathbf{v}_3.$$

The vector $\mathbf{v}_1$ was redundant, since

$$\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{Span}(\mathbf{v}_2, \mathbf{v}_3).$$

We might surmise that the vector $\mathbf{w}$ can now be written *uniquely* as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. This is easy to check with an augmented matrix:

$$\left[A' \mid \mathbf{w} \right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right];$$

from the fact that the matrix A' has rank 2, we infer that the system of equations has a unique solution. ▲

In the language of functions, the question of uniqueness is the question of whether the function $\mu_A : \mathbb{R}^3 \to \mathbb{R}^3$ is *one-to-one*. Remember that we say f is a *one-to-one* function if

$$\text{whenever } a \neq b, \text{ it must be the case that } f(a) \neq f(b).$$

Given some function $y = f(x)$, we might ask if, for a certain value r, we can solve the equation $f(x) = r$. When r is in the image of the function, there is at least one solution. Is the solution unique? If f is a one-to-one function, there can be *at most* one solution of the equation $f(x) = r$.

Next we show that the question of uniqueness we raised earlier can be reduced to one basic question, which will be crucial to all our future work.

Proposition 3.1. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. *If the zero vector has a* unique *expression as a linear combination of* $\mathbf{v}_1, \dots, \mathbf{v}_k$, *that is, if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \implies c_1 = c_2 = \cdots = c_k = 0,$$

then every *vector* $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ *has a* unique *expression as a linear combination of* $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Proof. By considering the matrix A whose column vectors are $\mathbf{v}_1, \dots, \mathbf{v}_k$, we can deduce this immediately from Proposition 5.4 of Chapter 1. However, we prefer to give a coordinate-free proof that is typical of many of the arguments we shall be encountering for a while.

Suppose that for some $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ there are two expressions

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \quad \text{and}$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k.$$

Then, subtracting, we obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + \cdots + (c_k - d_k)\mathbf{v}_k.$$

Since the only way to express the zero vector as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is with every coefficient equal to 0, we conclude that $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$, which means, of course, that $c_1 = d_1, c_2 = d_2, \dots, c_k = d_k$. That is, $\mathbf{v}$ has a unique expression as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$. □

This discussion leads us to make the following definition.

Definition. The (indexed) set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \implies c_1 = c_2 = \cdots = c_k = 0,$$

that is, if the *only* way of expressing the zero vector as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the *trivial* linear combination $0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k$.

The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called *linearly dependent* if it is not linearly independent—i.e., if there is some expression

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}, \quad \text{where not all the } c_i\text{'s are } 0.$$

The language is problematic here. Many mathematicians—including at least one of the authors of this text—often say things like "the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent." But linear independence (or dependence) is a property of the whole *collection* of vectors, not of the individual vectors. What's worse, we really should refer to an *ordered list* of vectors rather than to a set of vectors. For example, any list in which some vector, $\mathbf{v}$, appears twice is obviously giving a linearly dependent collection, but the set $\{\mathbf{v}, \mathbf{v}\}$ is indistinguishable from the set $\{\mathbf{v}\}$. There seems to be no ideal route out of this morass! Having said all this, we warn the gentle reader that we may occasionally say, "the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly (in)dependent" where it would be too clumsy to be more pedantic. Just stay alert!!

EXAMPLE 2

We wish to decide whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^4$$

form a linearly independent set.

Here is a piece of advice: It is virtually always the case that when you are presented with a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that you are to prove linearly independent, you should write,

"Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. I must show that $c_1 = \cdots = c_k = 0$."

You then use whatever hypotheses you're given to arrive at that conclusion.

The definition of linear independence is a particularly subtle one, largely because of the syntax. Suppose we know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. As a result, we know that *if* it should happen that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, *then* it must be that $c_1 = c_2 = \cdots = c_k = 0$. But we may never blithely assert that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$.

Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$, i.e.,

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}.$$

Can we conclude that $c_1 = c_2 = c_3 = 0$? We recognize this as a homogeneous system of linear equations:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}.$$

By now we are old hands at solving such systems. We find that the echelon form of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so our system of equations in fact has infinitely many solutions. For example, we can take $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. The vectors therefore form a linearly dependent set. ▲

EXAMPLE 3

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. We show next that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then so is $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$. Suppose

$$c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{v}+\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}) = \mathbf{0}.$$

We must show that $c_1 = c_2 = c_3 = 0$. We use the distributive property to rewrite our equation as

$$(c_1 + c_3)\mathbf{u} + (c_1 + c_2)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, we may infer that the coefficients of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ must each be equal to 0. Thus,

$$\begin{aligned} c_1 \qquad\quad + c_3 &= 0 \\ c_1 + c_2 \qquad\quad &= 0 \\ c_2 + c_3 &= 0, \end{aligned}$$

and we leave it to the reader to check that the only solution of this system of equations is, in fact, $c_1 = c_2 = c_3 = 0$, as desired. ▲

EXAMPLE 4

Any time one has a list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in which one of the vectors is the zero vector, say $\mathbf{v}_1 = \mathbf{0}$, then the set of vectors must be linearly dependent, because the equation

$$1\mathbf{v}_1 = \mathbf{0}$$

is a nontrivial linear combination of the vectors yielding the zero vector. ▲

EXAMPLE 5

How can two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ give rise to a linearly dependent set? By definition, this means that there is a linear combination

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0},$$

where, to start, *either* $a \neq 0$ *or* $b \neq 0$. But if, say, $a = 0$, then the equation reduces to $b\mathbf{v} = \mathbf{0}$; since $b \neq 0$, we must have $\mathbf{v} = \mathbf{0}$, which contradicts the hypothesis that the vectors are nonzero. Thus, in this case, we must have *both* a *and* $b \neq 0$. We may write $\mathbf{u} = -\frac{b}{a}\mathbf{v}$, so $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. Hence two nonzero linearly dependent vectors are parallel (and vice versa).

How can a collection of three nonzero vectors be linearly dependent? As before, there must be a linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0},$$

where (at least) one of a, b, and c is nonzero. Say $a \neq 0$. This means that we can solve

$$\mathbf{u} = -\frac{1}{a}(b\mathbf{v} + c\mathbf{w}) = \left(-\frac{b}{a}\right)\mathbf{v} + \left(-\frac{c}{a}\right)\mathbf{w},$$

so $\mathbf{u} \in \text{Span}\,(\mathbf{v}, \mathbf{w})$. In particular, $\text{Span}\,(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is either a line (if all three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are parallel) or a plane (when $\mathbf{v}$ and $\mathbf{w}$ are nonparallel). We leave it to the reader to think about what must happen when $a = 0$. ▲

The appropriate generalization of the last example is the following useful criterion, depicted in Figure 3.1.

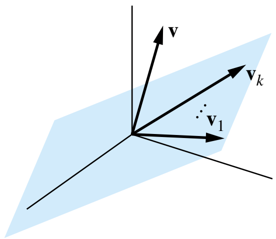


FIGURE 3.1

Proposition 3.2. *Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly independent set, and suppose $\mathbf{v} \in \mathbb{R}^n$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$.*

The *contrapositive* of the statement

"if P, then Q"

is

"if Q is false, then P is false."

One of the fundamental points of logic underlying all of mathematics is that these statements are equivalent: One is true precisely when the other is. (This is quite reasonable. For instance, if Q must be true whenever P is true and we know that Q is false, then P must be false as well, for if not, Q would have had to be true.)

It probably is a bit more convincing to consider a couple of examples:

- If we believe the statement "Whenever it is raining, the ground is wet" (or "*if* it is raining, *then* the ground is wet"), we should equally well grant that "If the ground is dry, then it is not raining."
- If we believe the statement "If $x = 2$, then $x^2 = 4$," then we should believe that "if $x^2 \neq 4$, then $x \neq 2$."

It is important not to confuse the contrapositive of a statement with the *converse* of the statement. The converse of the statement "if P, then Q" is

"if Q, then P."

Note that even if we believe our two earlier statements, we do *not* believe their converses:

- "If the ground is wet, then it is raining"—it may have stopped raining a while ago, or someone may have washed a car earlier.
- "If $x^2 = 4$, then $x = 2$"—even though this is a common error, it is an error nevertheless: x might be -2.

Proof. We will prove the contrapositive: Still supposing that $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly independent set,

$$\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\} \text{ is linearly dependent if and only if } \mathbf{v} \in \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

Suppose that $\mathbf{v} \in \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ for some scalars $c_1, \dots, c_k$, so

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + (-1)\mathbf{v} = \mathbf{0},$$

from which we conclude that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent (since at least one of the coefficients is nonzero).

Now suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent. This means that there are scalars $c_1, \dots, c_k$, and c, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + c\mathbf{v} = \mathbf{0}.$$

Note that we cannot have $c = 0$: For if c were 0, we'd have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, and linear independence of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ implies $c_1 = \cdots = c_k = 0$, which contradicts our assumption that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent. Therefore $c \neq 0$, and so

$$\mathbf{v} = -\frac{1}{c}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = \left(-\frac{c_1}{c}\right)\mathbf{v}_1 + \left(-\frac{c_2}{c}\right)\mathbf{v}_2 + \cdots + \left(-\frac{c_k}{c}\right)\mathbf{v}_k,$$

which tells us that $\mathbf{v} \in \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$, as required. □

We now understand that when we have a set of linearly independent vectors, no proper subset will yield the same span. In other words, we will have an "efficient" set of spanning vectors (that is, there is no redundancy in the vectors we've chosen; no proper subset will do). This motivates the following definition.

Definition. Let $V \subset \mathbb{R}^n$ be a subspace. The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called a *basis* for V if

(i) $\mathbf{v}_1, \dots, \mathbf{v}_k$ span V, that is, $V = \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$, and

(ii) $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

We comment that the plural of *basis* is *bases*.[4]

EXAMPLE 6

Let $\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1) \in \mathbb{R}^n$. Then $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, called the *standard basis*. To check this, we must establish that properties (i) and (ii) above hold for $V = \mathbb{R}^n$. The first is obvious: If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\mathbf{x} =$

[4] Pronounced *bā*s*ē*z, to rhyme with Macy's.

$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$. The second is not much harder. Suppose $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = \mathbf{0}$. This means that $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$, and so $c_1 = c_2 = \cdots = c_n = 0$. ▲

EXAMPLE 7

Consider the plane given by $V = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - x_2 + 2x_3 = 0\} \subset \mathbb{R}^3$. Our algorithms of Chapter 1 tell us that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

span V. Since these vectors are not parallel, it follows from Example 5 that they must be linearly independent.

For the practice, however, we give a direct argument. Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.$$

Writing out the entries explicitly, we obtain

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

from which we conclude that $c_1 = c_2 = 0$, as required. (For future reference, we note that this information came from the free variable "slots.") Therefore, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and gives a basis for V, as required. ▲

The following observation may prove useful.

Corollary 3.3. *Let $V \subset \mathbb{R}^n$ be a subspace, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for V if and only if every vector of V can be written uniquely as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.*

Proof. This is immediate from Proposition 3.1. □

This result is so important that we introduce a bit of terminology.

> *Definition.* When we write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$, we refer to $c_1, \ldots, c_k$ as the *coordinates* of $\mathbf{v}$ with respect to the (ordered) basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

EXAMPLE 8

Consider the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Let's take a general vector $\mathbf{b} \in \mathbb{R}^3$ and ask first of all whether it has a unique expression as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Forming the augmented matrix and row reducing, we find

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 1 & 0 & b_2 \\ 1 & 2 & 2 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2b_1 - b_3 \\ 0 & 1 & 0 & -4b_1 + b_2 + 2b_3 \\ 0 & 0 & 1 & 3b_1 - b_2 - b_3 \end{array}\right].$$

It follows from Corollary 3.3 that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, because an arbitrary vector $\mathbf{b} \in \mathbb{R}^3$ can be written in the form

$$\mathbf{b} = \underbrace{(2b_1 - b_3)}_{c_1}\mathbf{v}_1 + \underbrace{(-4b_1 + b_2 + 2b_3)}_{c_2}\mathbf{v}_2 + \underbrace{(3b_1 - b_2 - b_3)}_{c_3}\mathbf{v}_3.$$

And, what's more,

$$\begin{aligned} c_1 &= 2b_1 - b_3, \\ c_2 &= -4b_1 + b_2 + 2b_3, \quad \text{and} \\ c_3 &= 3b_1 - b_2 - b_3 \end{aligned}$$

give the coordinates of $\mathbf{b}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. ▲

Our experience in Example 8 leads us to make the following general observation:

Proposition 3.4. *Let A be an $n \times n$ matrix. Then A is nonsingular if and only if its column vectors form a basis for $\mathbb{R}^n$.*

Proof. As usual, let's denote the column vectors of A by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. Using Corollary 3.3, we are to prove that A is nonsingular if and only if every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. But this is exactly what Proposition 5.5 of Chapter 1 tells us. □

Somewhat more generally (see Exercise 12), we have the following result.

EXAMPLE 9

Suppose A is a nonsingular $n \times n$ matrix and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$. Then we wish to show that $\{A\mathbf{v}_1, \dots, A\mathbf{v}_n\}$ is likewise a basis for $\mathbb{R}^n$.

First, we show that $\{A\mathbf{v}_1, \dots, A\mathbf{v}_n\}$ is linearly independent. Following our ritual, we start by supposing that

$$c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n) = \mathbf{0},$$

and we wish to show that $c_1 = \cdots = c_n = 0$. By linearity properties we have

$$\begin{aligned} \mathbf{0} = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_nA\mathbf{v}_n &= A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + \cdots + A(c_n\mathbf{v}_n) \\ &= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n). \end{aligned}$$

Since A is nonsingular, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and so we must have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. From the linear independence of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ we now conclude that $c_1 = c_2 = \cdots = c_n = 0$, as required.

Now, why do these vectors span $\mathbb{R}^n$? (The result follows from Exercise 1.5.13, but we give the argument here.) Given $\mathbf{b} \in \mathbb{R}^n$, we know from Proposition 5.5 of Chapter 1 that there is a unique $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$, we can write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ for some scalars $c_1, \ldots, c_n$. Then, again by linearity properties, we have

$$\begin{aligned}\mathbf{b} = A\mathbf{x} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) &= A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + \cdots + A(c_n\mathbf{v}_n) \\ &= c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n),\end{aligned}$$

as required. ▲

Given a subspace $V \subset \mathbb{R}^n$, how do we know there is some basis for it? This is a consequence of Proposition 3.2 as well.

Theorem 3.5. *Any subspace $V \subset \mathbb{R}^n$ other than the trivial subspace has a basis.*

Proof. Because $V \neq \{\mathbf{0}\}$, we can choose a nonzero vector $\mathbf{v}_1 \in V$. If $\mathbf{v}_1$ spans V, then we know $\{\mathbf{v}_1\}$ will constitute a basis for V. If not, choose $\mathbf{v}_2 \notin \text{Span}(\mathbf{v}_1)$. From Proposition 3.2 we infer that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. If $\mathbf{v}_1, \mathbf{v}_2$ span V, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ will be a basis for V. If not, choose $\mathbf{v}_3 \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Once again, we know that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be linearly independent and hence will form a basis for V if the three vectors span V. We continue in this fashion, and we are guaranteed that the process will terminate in at most n steps: Once we have $n+1$ vectors in $\mathbb{R}^n$, they must form a linearly dependent set, because an $n \times (n+1)$ matrix has rank at most n (see Exercise 15). □

From this fact it follows that every subspace $V \subset \mathbb{R}^n$ can be expressed as the row space (or column space) of a matrix. This settles the issue raised in the footnote on p. 137. As an application, we can now follow through on the substance of the remark on p. 140.

Proposition 3.6. *Let $V \subset \mathbb{R}^n$ be a subspace. Then $(V^\perp)^\perp = V$.*

Proof. Choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for V, and consider the $k \times n$ matrix A whose rows are $\mathbf{v}_1, \ldots, \mathbf{v}_k$. By construction, $V = \mathbf{R}(A)$. By Theorem 2.5, $V^\perp = \mathbf{R}(A)^\perp = \mathbf{N}(A)$, and $\mathbf{N}(A)^\perp = \mathbf{R}(A)$, so $(V^\perp)^\perp = V$. □

We conclude this section with the problem of determining bases for each of the four fundamental subspaces of a matrix.

EXAMPLE 10

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}.$$

Gaussian elimination gives us the reduced echelon form R:

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this information, we wish to find bases for $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^\top)$.

Since any row of R is a linear combination of rows of A and vice versa, it is easy to see that $\mathbf{R}(A) = \mathbf{R}(R)$ (see Exercise 3.2.1), so we concentrate on the rows of R. We may as well use only the nonzero rows of R; now we need only check that they form a linearly independent set. We keep an eye on the pivot "slots": Suppose

$$c_1 \begin{bmatrix} \textcircled{1} \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \textcircled{1} \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \textcircled{1} \\ 1 \end{bmatrix} = \mathbf{0}.$$

This means that

$$\begin{bmatrix} c_1 \\ c_2 \\ -c_1 + c_2 \\ c_3 \\ c_1 + 2c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and so $c_1 = c_2 = c_3 = 0$, as promised.

From the reduced echelon form R, we read off the vectors that span $\mathbf{N}(A)$: The general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_3 - x_5 \\ -x_3 - 2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

span $\mathbf{N}(A)$. On the other hand, these vectors are linearly independent, because if we take a linear combination

$$x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \mathbf{0},$$

we infer (from the free variable slots) that $x_3 = x_5 = 0$.

Obviously, $\mathbf{C}(A)$ is spanned by the five column vectors of A. But these vectors cannot be linearly independent—that's what vectors in the nullspace of A tell us. From our vectors spanning $\mathbf{N}(A)$, we know that

$$(*) \qquad \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \quad \text{and} \quad -\mathbf{a}_1 - 2\mathbf{a}_2 - \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{0}.$$

These equations tell us that $\mathbf{a}_3$ and $\mathbf{a}_5$ can be written as linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$. If we can check that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly independent, we'll be finished. So we form a matrix A' with these columns (easier: cross out the third and fifth columns of A), and reduce it to echelon form (easier yet: cross out the third and fifth columns of R). Well, we have

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R',$$

and so only the trivial linear combination of the columns of A' will yield the zero vector. In conclusion, the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

give a basis for $\mathbf{C}(A)$.

Remark. The puzzled reader may wonder why, looking at the equations (∗), we chose to use the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$ and discard the vectors $\mathbf{a}_3$ and $\mathbf{a}_5$. These are the columns in which pivots appear in the echelon form; the subsequent reasoning establishes their linear independence. There might in any specific case be other viable choices for vectors to discard, but then the proof that the remaining vectors form a linearly independent set may be less straightforward.

What about the left nullspace? The only row of 0's in R arises as the linear combination

$$-\mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 = \mathbf{0}$$

of the rows of A, so we expect the vector

$$\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

to give a basis for $\mathbf{N}(A^{\mathsf{T}})$. As a check, we note it is orthogonal to the basis vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$ for $\mathbf{C}(A)$. Could there be any vectors in $\mathbf{C}(A)^{\perp}$ besides multiples of $\mathbf{v}$? ▲

What is lurking in the background here is a notion of dimension, and we turn to this important topic in the next section.

Exercises 3.3

1. Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (2, 4, 5)$, and $\mathbf{v}_3 = (2, 4, 6) \in \mathbb{R}^3$. Is each of the following statements correct or incorrect? Explain.
   a. The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.
   b. Each of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ can be written as a linear combination of the others.

*2. Decide whether each of the following sets of vectors is linearly independent.

a. $\{(1, 4), (2, 9)\} \subset \mathbb{R}^2$

b. $\{(1, 4, 0), (2, 9, 0)\} \subset \mathbb{R}^3$

c. $\{(1, 4, 0), (2, 9, 0), (3, -2, 0)\} \subset \mathbb{R}^3$

d. $\{(1, 1, 1), (2, 3, 3), (0, 1, 2)\} \subset \mathbb{R}^3$

e. $\{(1, 1, 1, 3), (1, 1, 3, 1), (1, 3, 1, 1), (3, 1, 1, 1)\} \subset \mathbb{R}^4$

f. $\{(1, 1, 1, -3), (1, 1, -3, 1), (1, -3, 1, 1), (-3, 1, 1, 1)\} \subset \mathbb{R}^4$

*3. Decide whether the following sets of vectors give a basis for the indicated space.

a. $\{(1, 2, 1), (2, 4, 5), (1, 2, 3)\}$; $\mathbb{R}^3$

b. $\{(1, 0, 1), (1, 2, 4), (2, 2, 5), (2, 2, -1)\}$; $\mathbb{R}^3$

c. $\{(1, 0, 2, 3), (0, 1, 1, 1), (1, 1, 4, 4)\}$; $\mathbb{R}^4$

d. $\{(1, 0, 2, 3), (0, 1, 1, 1), (1, 1, 4, 4), (2, -2, 1, 2)\}$; $\mathbb{R}^4$

4. In each case, check that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ and give the coordinates of the given vector $\mathbf{b} \in \mathbb{R}^n$ with respect to that basis.

a. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

*b. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

c. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

*d. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

5. Following Example 10, for each of the following matrices A, give a basis for each of the subspaces $\mathbf{R}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A)$, and $\mathbf{N}(A^{\mathsf{T}})$.

*a. $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix}$

*c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -1 & -1 \end{bmatrix}$

*6. Give a basis for the orthogonal complement of the subspace $W \subset \mathbb{R}^4$ spanned by $(1, 1, 1, 2)$ and $(1, -1, 5, 2)$.

7. Let $V \subset \mathbb{R}^5$ be spanned by $(1, 0, 1, 1, 1)$ and $(0, 1, -1, 0, 2)$. By finding the left nullspace of an appropriate matrix, give a homogeneous system of equations having V as its solution set. Explain how you are using Proposition 3.6.

8. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. Prove that $\{\mathbf{v} - \mathbf{w},\ 2\mathbf{v} + \mathbf{w}\}$ is linearly independent as well.

9. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ form a linearly independent set. Prove that $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + 2\mathbf{w}$, and $-\mathbf{u} + \mathbf{v} + \mathbf{w}$ likewise form a linearly independent set.

♯**10.** Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ are nonzero vectors with the property that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent. (*Hint:* "Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$." Start by showing $c_1 = 0$.)

♯**11.** Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ are nonzero, mutually orthogonal vectors in $\mathbb{R}^n$.

a. Prove that they form a basis for $\mathbb{R}^n$. (Use Exercise 10.)

b. Given any $\mathbf{x} \in \mathbb{R}^n$, give an explicit formula for the coordinates of $\mathbf{x}$ with respect to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

c. Deduce from your answer to part *b* that $\mathbf{x} = \sum_{i=1}^{n} \text{proj}_{\mathbf{v}_i}\mathbf{x}$.

12. Give an alternative proof of Example 9 by applying Proposition 3.4 and Exercise 2.1.10.

***13.** Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent, then every vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$ infinitely many ways.

♯***14.** Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly independent set. Show that for any $1 \leq \ell < k$, the set $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is linearly independent as well.

♯**15.** Suppose $k > n$. Prove that any k vectors in $\mathbb{R}^n$ must form a linearly dependent set. (So what can you conclude if you have k linearly independent vectors in $\mathbb{R}^n$?)

16. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly dependent set. Prove that for some j between 1 and k we have $\mathbf{v}_j \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$. That is, one of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ can be written as a linear combination of the remaining vectors.

17. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly dependent set. Prove that either $\mathbf{v}_1 = \mathbf{0}$ or $\mathbf{v}_i \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1})$ for some $i = 2, 3, \dots, k$. (*Hint:* There is a relation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ with at least one $c_j \neq 0$. Consider the largest such j.)

18. Let A be an $m \times n$ matrix and suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. Prove that if $\{A\mathbf{v}_1, \dots, A\mathbf{v}_k\}$ is linearly independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ must be linearly independent.

19. Let A be an $n \times n$ matrix. Prove that if A is nonsingular and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, then $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_k\}$ is likewise linearly independent. Give an example to show that the result is false if A is singular.

20. Suppose U and V are subspaces of $\mathbb{R}^n$. Prove that $(U \cap V)^\perp = U^\perp + V^\perp$. (*Hint:* Use Exercise 3.1.18 and Proposition 3.6.)

♯**21.** Let A be an $m \times n$ matrix of rank n. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent. Prove that $\{A\mathbf{v}_1, \dots, A\mathbf{v}_k\} \subset \mathbb{R}^m$ is likewise linearly independent. (**N.B.**: If you did not explicitly make use of the assumption that $\text{rank}(A) = n$, your proof cannot be correct. Why?)

22. Let A be an $n \times n$ matrix and suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ are nonzero vectors that satisfy

$$\begin{aligned} A\mathbf{v}_1 &= \mathbf{v}_1 \\ A\mathbf{v}_2 &= 2\mathbf{v}_2 \\ A\mathbf{v}_3 &= 3\mathbf{v}_3. \end{aligned}$$

Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (*Hint:* Start by showing that $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent.)

***23.** Suppose U and V are subspaces of $\mathbb{R}^n$ with $U \cap V = \{\mathbf{0}\}$. If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis for U and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is a basis for V, prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is a basis for $U + V$.

4 Dimension and Its Consequences

Once we realize that every subspace $V \subset \mathbb{R}^n$ has *some* basis, we are confronted with the problem that it has *many* of them. For example, Proposition 3.4 gives us a way of finding zillions of bases for $\mathbb{R}^n$. As we shall now show, all bases for a given subspace have one thing in common: They all consist of the same number of elements. To establish this, we begin by proving an appropriate generalization of Exercise 3.3.15.

Proposition 4.1. *Let $V \subset \mathbb{R}^n$ be a subspace, let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for V, and let $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in V$. If $\ell > k$, then $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ must be linearly dependent.*

Proof. Each vector in V can be written uniquely as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$. So let's write each vector $\mathbf{w}_1, \dots, \mathbf{w}_\ell$ as such:

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \dots + a_{k1}\mathbf{v}_k \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \dots + a_{k2}\mathbf{v}_k \\
&\ \ \vdots \\
\mathbf{w}_\ell &= a_{1\ell}\mathbf{v}_1 + a_{2\ell}\mathbf{v}_2 + \dots + a_{k\ell}\mathbf{v}_k.
\end{aligned}$$

Then we can write

$$(*) \qquad \begin{bmatrix} | & & | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_j & \cdots & \mathbf{w}_\ell \\ | & & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & & a_{1j} & & a_{1\ell} \\ a_{21} & & a_{2j} & & a_{2\ell} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{k1} & & a_{kj} & & a_{k\ell} \end{bmatrix}}_{A},$$

where the j^{th} *column* of the $k \times \ell$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ consists of the coordinates of the vector $\mathbf{w}_j$ with respect to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. We can rewrite $(*)$ as

$$(**) \qquad \begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_\ell \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} A.$$

Since $\ell > k$, there cannot be a pivot in every column of A, and so there is a *nonzero* vector $\mathbf{c}$ satisfying

$$A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} = \mathbf{0}.$$

Using (∗∗) and associativity, we have

$$\begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_\ell \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \left(A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} \right) = \mathbf{0}.$$

That is, we have found a nontrivial linear combination

$$c_1\mathbf{w}_1 + \cdots + c_\ell\mathbf{w}_\ell = \mathbf{0},$$

which means that $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is linearly dependent, as was claimed. □

(See Exercise 15 for an analogous result related to spanning sets.)

Proposition 4.1 leads directly to our main result.

Theorem 4.2. *Let $V \subset \mathbb{R}^n$ be a subspace, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ be two bases for V. Then we have $k = \ell$.*

Proof. Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ forms a basis for V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is known to be linearly independent, we use Proposition 4.1 to conclude that $\ell \le k$. Now here's the trick: $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is likewise a basis for V, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is known to be linearly independent, so we infer from Proposition 4.1 that $k \le \ell$. The only way both inequalities can hold is for k and ℓ to be equal, as we wished to show. □

> This is the numerical analogue of our common practice of proving two sets are equal by showing that each is a subset of the other. Here we show two numbers are equal by showing that each is less than or equal to the other.

We now make the official definition.

> ***Definition.*** The *dimension* of a subspace $V \subset \mathbb{R}^n$ is the number of vectors in any basis for V. We denote the dimension of V by $\dim V$. By convention, $\dim\{\mathbf{0}\} = 0$.

EXAMPLE 1

By virtue of Example 6 in Section 3, the dimension of $\mathbb{R}^n$ itself is n. A line through the origin is a one-dimensional subspace, and a plane through the origin is a two-dimensional subspace. ▲

As we shall see in our applications, dimension is a powerful tool. Here is the first instance.

Proposition 4.3. *Suppose V and W are subspaces of $\mathbb{R}^n$ with the property that $W \subset V$. If $\dim V = \dim W$, then $V = W$.*

Proof. Let $\dim W = k$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for W. If $W \subsetneq V$, then there must be a vector $\mathbf{v} \in V$ with $\mathbf{v} \notin W$. From Proposition 3.2 we infer that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is linearly independent, so $\dim V \ge k + 1$. This is a contradiction. Therefore, $V = W$. □

The next result is also quite useful.

Proposition 4.4. *Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then any k vectors that span V must be linearly independent, and any k linearly independent vectors in V must span V.*

Proof. Left to the reader in Exercise 16. □

We now turn to a few explicit examples of finding bases for subspaces.

EXAMPLE 2

Let $V = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \subset \mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}.$$

We would like to determine whether some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ gives a basis for V. Of course, this set of four vectors must be linearly dependent, since $V \subset \mathbb{R}^3$ and $\mathbb{R}^3$ is only three-dimensional. Now let's examine the solutions of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0},$$

or, in matrix form,

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathbf{0}.$$

As usual, we proceed to reduced echelon form:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

from which we find that the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

span the space of solutions. In particular, this tells us that

$$-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \qquad \text{and} \qquad -3\mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0},$$

and so the vectors $\mathbf{v}_2$ and $\mathbf{v}_4$ can be expressed as linear combinations of the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$. On the other hand, $\{\mathbf{v}_1, \mathbf{v}_3\}$ is linearly independent (why?), so this gives a basis for V. ▲

EXAMPLE 3

Given linearly independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} \in \mathbb{R}^4,$$

we wish to find additional vectors $\mathbf{v}_3, \mathbf{v}_4, \dots$ to make up a basis for $\mathbb{R}^4$ (as we shall see in Exercise 17, this is always possible). First of all, since $\dim \mathbb{R}^4 = 4$, we know that only two further vectors will be required. How should we find them? We might try guessing; a more methodical approach is suggested in Exercise 2. But here we try a more geometric solution.

Let's find a basis for the orthogonal complement of the subspace V spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. We consider the matrix A whose *row* vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}.$$

The vectors

$$\mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbf{N}(A)$ (and therefore form a basis for $\mathbf{N}(A)$—why?). By Proposition 2.2, $\{\mathbf{v}_3, \mathbf{v}_4\}$ gives a basis for $\mathbf{R}(A)^\perp = V^\perp$. We now give a geometric argument that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ forms a basis for $\mathbb{R}^4$. (Alternatively, the reader could just check this in a straightforward numerical fashion.)

By Proposition 4.4, we need only check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Suppose, as usual, that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

This means that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = -(c_3\mathbf{v}_3 + c_4\mathbf{v}_4),$$

but the vector on the left hand side is in V and the vector on the right hand side is in $V^\perp$. By Exercise 3.1.10, only $\mathbf{0}$ can lie in both V and $V^\perp$, and so we conclude that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \qquad \text{and} \qquad c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

Now from the fact that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, we infer that $c_1 = c_2 = 0$, and from the fact that $\{\mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent, we infer that $c_3 = c_4 = 0$. In sum, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent and therefore gives a basis for $\mathbb{R}^4$. ▲

4.1 Back to the Four Fundamental Subspaces

We now return to the four fundamental subspaces associated to any matrix. We specify a procedure for giving a basis for each of $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^\mathsf{T})$. Their dimensions will follow immediately. It may help the reader to compare this discussion with Example 10 in Section 3; indeed, the theorem is best understood by working several examples.

Theorem 4.5. *Let A be an $m \times n$ matrix. Let U and R, respectively, denote the echelon and reduced echelon form, respectively, of A, and write $EA = U$ (so E is the product of the elementary matrices by which we reduce A to some echelon form).*

1. *The (transposes of the) nonzero rows of U (or of R) give a basis for $\mathbf{R}(A)$.*
2. *The vectors obtained by setting each free variable equal to 1 and the remaining free variables equal to 0 in the general solution of $A\mathbf{x} = \mathbf{0}$ (which we read off from $R\mathbf{x} = \mathbf{0}$) give a basis for $\mathbf{N}(A)$.*

3. *The pivot columns of A (i.e., the columns of the original matrix A corresponding to the pivots in U) give a basis for $\mathbf{C}(A)$.*
4. *The (transposes of the) rows of E that correspond to the zero rows of U give a basis for $\mathbf{N}(A^{\mathsf{T}})$. (The same works with E' if we write $E'A = R$.)*

Proof. For simplicity of exposition, let's assume that the reduced echelon form takes the shape

$$R = \begin{array}{c} r\left\{ \vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}} \right. \\ m-r\left\{ \vphantom{O} \right. \end{array} \left[\begin{array}{ccc|cccc} 1 & & & b_{1,r+1} & b_{1,r+2} & \cdots & b_{1n} \\ & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & 1 & b_{r,r+1} & b_{r,r+2} & \cdots & b_{rn} \\ \hline & O & & & O & & \end{array}\right].$$

1. Since row operations are invertible, $\mathbf{R}(A) = \mathbf{R}(U)$ (see Exercise 3.2.1). Clearly the nonzero rows of U span $\mathbf{R}(U)$. Moreover, they are linearly independent because of the pivots. Let $\mathbf{U}_1, \dots, \mathbf{U}_r$ denote the nonzero rows of U; because of our simplifying assumption on R, we know that the pivots of U occur in the first r columns as well. Suppose now that

$$c_1\mathbf{U}_1 + \cdots + c_r\mathbf{U}_r = \mathbf{0}.$$

The first entry of the left-hand side is c_1u_{11} (since the first entry of the vectors $\mathbf{U}_2, \dots, \mathbf{U}_r$ is 0 by definition of echelon form). Because $u_{11} \neq 0$ by definition of pivot, we must have $c_1 = 0$. Continuing in this fashion, we find that $c_1 = c_2 = \cdots = c_r = 0$. In conclusion, $\{\mathbf{U}_1, \dots, \mathbf{U}_r\}$ forms a basis for $\mathbf{R}(U)$ and hence for $\mathbf{R}(A)$.

2. $A\mathbf{x} = \mathbf{0}$ if and only if $R\mathbf{x} = \mathbf{0}$, which means that

$$\begin{array}{ccccccccccccc} x_1 & & & & + & b_{1,r+1}x_{r+1} & + & b_{1,r+2}x_{r+2} & + & \cdots & + & b_{1n}x_n & = 0 \\ & x_2 & & & + & b_{2,r+1}x_{r+1} & + & b_{2,r+2}x_{r+2} & + & \cdots & + & b_{2n}x_n & = 0 \\ & & \ddots & & & \vdots & & \vdots & & & & \vdots & \vdots \\ & & & x_r & + & b_{r,r+1}x_{r+1} & + & b_{r,r+2}x_{r+2} & + & \cdots & + & b_{rn}x_n & = 0. \end{array}$$

Thus, an arbitrary element of $\mathbf{N}(A)$ can be written in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = x_{r+1}\begin{bmatrix} -b_{1,r+1} \\ \vdots \\ -b_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2}\begin{bmatrix} -b_{1,r+2} \\ \vdots \\ -b_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} -b_{1n} \\ \vdots \\ -b_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The assertion is then that the vectors

$$\begin{bmatrix} -b_{1,r+1} \\ \vdots \\ -b_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -b_{1,r+2} \\ \vdots \\ -b_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} -b_{1n} \\ \vdots \\ -b_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

give a basis for $\mathbf{N}(A)$. They obviously span $\mathbf{N}(A)$, because every vector in $\mathbf{N}(A)$ can be expressed as a linear combination of them. We need to check linear independence: The key is the pattern of 1's and 0's in the free-variable "slots." Suppose

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x_{r+1} \begin{bmatrix} -b_{1,r+1} \\ \vdots \\ -b_{r,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2} \begin{bmatrix} -b_{1,r+2} \\ \vdots \\ -b_{r,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} -b_{1n} \\ \vdots \\ -b_{rn} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Then we get $x_{r+1} = x_{r+2} = \dots = x_n = 0$, as required.

3. Let's continue with the notational simplification that the pivots occur in the first r columns. Then we need to establish the fact that the first r column vectors of the *original* matrix A give a basis for $\mathbf{C}(A)$. These vectors form a linearly independent set, since the only solution of

$$c_1\mathbf{a}_1 + \dots + c_r\mathbf{a}_r = \mathbf{0}$$

is $c_1 = c_2 = \dots = c_r = 0$ (look only at the first r columns of A and the first r columns of R). It is more interesting to understand why $\mathbf{a}_1, \dots, \mathbf{a}_r$ span $\mathbf{C}(A)$. Consider each of the basis vectors for $\mathbf{N}(A)$ given above: Each one gives us a linear combination of the column vectors of A that results in the zero vector. In particular, we find that

$$\begin{array}{rcccccccl} -b_{1,r+1}\mathbf{a}_1 & - & \cdots & - & b_{r,r+1}\mathbf{a}_r & + & \mathbf{a}_{r+1} & & = \mathbf{0} \\ -b_{1,r+2}\mathbf{a}_1 & - & \cdots & - & b_{r,r+2}\mathbf{a}_r & & + \ \mathbf{a}_{r+2} & & = \mathbf{0} \\ \vdots & & & & \vdots & & & \ddots & \vdots \\ -b_{1n}\mathbf{a}_1 & - & \cdots & - & b_{rn}\mathbf{a}_r & & & + \ \mathbf{a}_n & = \mathbf{0}, \end{array}$$

from which we conclude that the vectors $\mathbf{a}_{r+1}, \dots, \mathbf{a}_n$ are all linear combinations of $\mathbf{a}_1, \dots, \mathbf{a}_r$. It follows that $\mathbf{C}(A)$ is spanned by $\mathbf{a}_1, \dots, \mathbf{a}_r$, as required.

4. Recall that vectors in $\mathbf{N}(A^{\mathsf{T}})$ correspond to ways of expressing the zero vector as linear combinations of the *rows* of A. The first r rows of the echelon matrix

U form a linearly independent set, whereas the last $m - r$ rows of U consist just of $\mathbf{0}$. Thus, we conclude from $EA = U$ that the (transposes of the) last $m - r$ rows of E span $\mathbf{N}(A^\top)$.[5] But these vectors are linearly independent, because E is nonsingular. Thus, the vectors $\mathbf{E}_{r+1}, \dots, \mathbf{E}_m$ give a basis for $\mathbf{N}(A^\top)$. □

Remark. Referring to our earlier discussion of (†) on p. 114 and our discussion in Sections 2 and 3 of this chapter, we finally know that finding the constraint equations for $\mathbf{C}(A)$ will give a *basis* for $\mathbf{N}(A^\top)$. It is also worth noting that to find bases for the four fundamental subspaces of the matrix A, we need only find the echelon form of A to deal with $\mathbf{R}(A)$ and $\mathbf{C}(A)$, the reduced echelon form of A to deal with $\mathbf{N}(A)$, and the echelon form of the augmented matrix $[\,A \mid \mathbf{b}\,]$ to deal with $\mathbf{N}(A^\top)$.

EXAMPLE 4

We want bases for $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^\top)$, given the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & -1 & -3 \end{bmatrix}.$$

We leave it to the reader to check that the reduced echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and that $EA = U$, where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Alternatively, the echelon form of the augmented matrix $[\,A \mid \mathbf{b}\,]$ is

$$[\,EA \mid E\mathbf{b}\,] = \left[\begin{array}{ccccc|l} 1 & 1 & 2 & 0 & 0 & b_1 \\ 0 & 1 & 1 & -1 & -1 & b_2 \\ 0 & 0 & 0 & 1 & 2 & -b_1 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -4b_1 + b_2 + 2b_3 + b_4 \end{array}\right].$$

[5] A vector not in their span would then give rise to a relation among the *nonzero* rows of U, which we know to be linearly independent. A formal argument comes from writing $A = E^{-1}U$, so $A^\top\mathbf{x} = \mathbf{0}$ if and only if $U^\top\big((E^\top)^{-1}\mathbf{x}\big) = \mathbf{0}$, and this happens if and only if $\mathbf{x} = E^\top\mathbf{y}$ for some $\mathbf{y} \in \mathbf{N}(U^\top)$. Since $\mathbf{N}(U^\top)$ is spanned by the last $m - r$ standard basis vectors for $\mathbb{R}^m$, our claim follows.

Applying Theorem 4.5, we obtain the following bases for the respective subspaces:

$$\mathbf{R}(A): \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad \text{(using } U\text{)}$$

$$\mathbf{N}(A): \quad \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \qquad \text{(using } R \text{ as in Chapter 1)}$$

$$\mathbf{C}(A): \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\} \qquad \text{(using the pivot columns of } A\text{)}$$

$$\mathbf{N}(A^{\mathsf{T}}): \quad \left\{ \begin{bmatrix} -4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad \text{(using the bottom row of } E\text{)}$$

The reader should check these all carefully. Note that we have $\dim \mathbf{R}(A) = \dim \mathbf{C}(A) = 3$, $\dim \mathbf{N}(A) = 2$, and $\dim \mathbf{N}(A^{\mathsf{T}}) = 1$. ▲

4.2 Important Consequences

We now deduce the following results on dimension. Recall that the rank of a matrix is the number of pivots in its echelon form.

Theorem 4.6. *Let A be an $m \times n$ matrix of rank r. Then*

1. $\dim \mathbf{R}(A) = \dim \mathbf{C}(A) = r$.
2. $\dim \mathbf{N}(A) = n - r$.
3. $\dim \mathbf{N}(A^{\mathsf{T}}) = m - r$.

Proof. There are r pivots and a pivot in each nonzero row of U, so $\dim \mathbf{R}(A) = r$. Similarly, we have a basis vector for $\mathbf{C}(A)$ for every pivot, so $\dim \mathbf{C}(A) = r$, as well. We see that $\dim \mathbf{N}(A)$ is equal to the number of free variables, and this is the difference between the total number of variables, n, and the number of pivot variables, r. Last, the number of zero rows in U is the difference between the total number of rows, m, and the number of nonzero rows, r, so $\dim \mathbf{N}(A^{\mathsf{T}}) = m - r$. □

An immediate corollary of Theorem 4.6 is the following. The dimension of the nullspace of A is often called the *nullity* of A, which is denoted null (A).

Corollary 4.7 (Nullity-Rank Theorem). *Let A be an $m \times n$ matrix. Then*

$$\text{null}\,(A) + \text{rank}(A) = n.$$

Now we are in a position to complete our discussion of orthogonal complements.

Proposition 4.8. *Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then* $\dim V^{\perp} = n - k$.

Proof. Choose a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for V, and let these be the *rows* of a $k \times n$ matrix A. By construction, we have $\mathbf{R}(A) = V$. Notice also that $\text{rank}(A) = \dim \mathbf{R}(A) = \dim V = k$. By Proposition 2.2, we have $V^{\perp} = \mathbf{N}(A)$, so $\dim V^{\perp} = \dim \mathbf{N}(A) = n - k$. □

As a consequence, we can prove a statement that should have been quite plausible all along (see Figure 4.1).

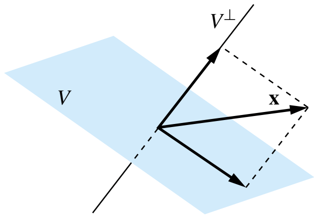


FIGURE 4.1

Theorem 4.9. *Let $V \subset \mathbb{R}^n$ be a subspace. Then every vector in $\mathbb{R}^n$ can be written uniquely as the sum of a vector in V and a vector in $V^{\perp}$. In particular, we have $\mathbb{R}^n = V + V^{\perp}$.*

Proof. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for V and $\{\mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ be a basis for $V^{\perp}$; note that we are using the result of Proposition 4.8 here. Then we claim that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent. For suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Then we have

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k}_{\text{element of } V} = \underbrace{-\left(c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n\right)}_{\text{element of } V^{\perp}}.$$

Because only the zero vector can be in both V and $V^{\perp}$ (for any such vector must be orthogonal to itself and therefore have length 0), we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{and} \quad c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since each of the sets $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\{\mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ is linearly independent, we conclude that $c_1 = \cdots = c_k = c_{k+1} = \cdots = c_n = 0$, as required.

It now follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ gives a basis for an n-dimensional subspace of $\mathbb{R}^n$, which by Proposition 4.3 must be all of $\mathbb{R}^n$. Thus, every vector $\mathbf{x} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{x} = \underbrace{\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k\right)}_{\text{element of } V} + \underbrace{\left(c_{k+1}\mathbf{v}_{k+1} + \cdots + c_n\mathbf{v}_n\right)}_{\text{element of } V^{\perp}},$$

as desired. □

Although in Chapter 4 we shall learn a better way to so decompose a vector, an example is instructive.

EXAMPLE 5

Let $V = \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \subset \mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Given an arbitrary $\mathbf{b} \in \mathbb{R}^4$, we wish to express $\mathbf{b}$ as the sum of a vector in V and a vector in $V^\perp$. Letting

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

we see that $V = \mathbf{R}(A)$, and so, by Proposition 2.2, we have $V^\perp = \mathbf{N}(A)$, for which the vectors

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

give a basis.

To give the coordinates of $\mathbf{b} \in \mathbb{R}^4$ with respect to the basis, we can use Gaussian elimination:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & b_1 \\ 0 & 1 & -1 & -1 & b_2 \\ 1 & 1 & 1 & 0 & b_3 \\ -1 & 1 & 0 & 1 & b_4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{3}(b_1 - b_3 + b_4) \\ 0 & 1 & 0 & 0 & \frac{1}{3}(b_2 + b_3 + b_4) \\ 0 & 0 & 1 & 0 & \frac{1}{3}(-b_1 - b_2 + b_3) \\ 0 & 0 & 0 & 1 & \frac{1}{3}(b_1 - b_2 + b_4) \end{array}\right].$$

Thus,

$$\mathbf{b} = \underbrace{\tfrac{1}{3}(b_1 - b_3 + b_4)\mathbf{v}_1 + \tfrac{1}{3}(b_2 + b_3 + b_4)\mathbf{v}_2}_{\in V} + \underbrace{\tfrac{1}{3}(-b_1 - b_2 + b_3)\mathbf{v}_3 + \tfrac{1}{3}(b_1 - b_2 + b_4)\mathbf{v}_4}_{\in V^\perp},$$

as required. (In this case, there is a foxier way to arrive at the answer. Because $\mathbf{v}_1, \dots, \mathbf{v}_4$ are mutually orthogonal, if $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$, then we can dot this equation with $\mathbf{v}_i$ to obtain $c_i = \mathbf{b} \cdot \mathbf{v}_i / \|\mathbf{v}_i\|^2$.) ▲

To close our discussion now, we recall in Figure 4.2 the schematic diagram given in Section 3 summarizing the geometric relation among our four fundamental subspaces. It follows from Theorem 2.5 and Theorem 4.9 that for any $m \times n$ matrix A, we have $\mathbf{R}(A) + \mathbf{N}(A) = \mathbb{R}^n$ and $\mathbf{C}(A) + \mathbf{N}(A^\mathsf{T}) = \mathbb{R}^m$. Now we will elaborate by considering the roles of the linear maps μ_A and μ_{A^T}. Recall that we have

$$\begin{aligned} \mu_A\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad & \mu_A(\mathbf{x}) = A\mathbf{x} \\ \mu_{A^\mathsf{T}}\colon \mathbb{R}^m \to \mathbb{R}^n, \qquad & \mu_{A^\mathsf{T}}(\mathbf{y}) = A^\mathsf{T}\mathbf{y}. \end{aligned}$$

μ_A sends all of $\mathbf{N}(A)$ to $\mathbf{0} \in \mathbb{R}^m$ and μ_{A^T} sends all of $\mathbf{N}(A^\mathsf{T})$ to $\mathbf{0} \in \mathbb{R}^n$. Now, the column space of A consists of all vectors of the form $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$; that is, it is the image of the function μ_A. Since $\dim \mathbf{R}(A) = \dim \mathbf{C}(A)$, this suggests that μ_A maps the subspace

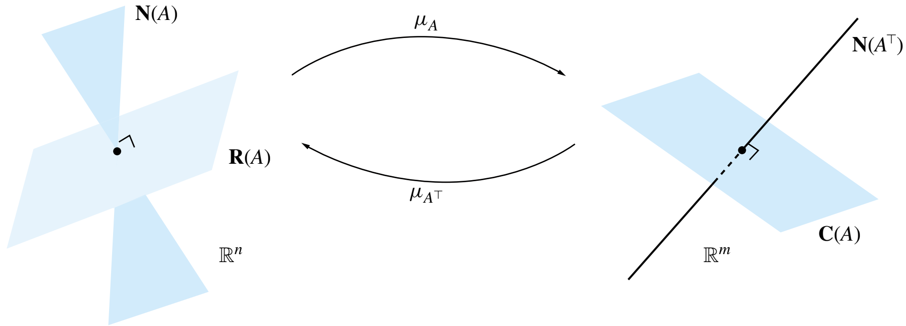


FIGURE 4.2

$\mathbf{R}(A)$ one-to-one and onto $\mathbf{C}(A)$. (And, symmetrically, $\mu_{A^\top}$ maps $\mathbf{C}(A)$ one-to-one and onto $\mathbf{R}(A)$.[6])

Proposition 4.10. *For each* $\mathbf{b} \in \mathbf{C}(A)$*, there is a unique vector* $\mathbf{x}_0 \in \mathbf{R}(A)$ *such that* $A\mathbf{x}_0 = \mathbf{b}$.

Proof. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a basis for $\mathbf{R}(A)$. Then $A\mathbf{v}_1, \dots, A\mathbf{v}_r$ are r vectors in $\mathbf{C}(A)$. They are linearly independent (by a modification of the proof of Exercise 3.3.21 that we leave to the reader). Therefore, by Proposition 4.4, these vectors must span $\mathbf{C}(A)$. This tells us that every vector $\mathbf{b} \in \mathbf{C}(A)$ is of the form $\mathbf{b} = A\mathbf{x}_0$ for some $\mathbf{x}_0 \in \mathbf{R}(A)$ (why?). And there can be only one such vector $\mathbf{x}_0$ because $\mathbf{R}(A) \cap \mathbf{N}(A) = \{\mathbf{0}\}$ (See Figure 4.3.) □

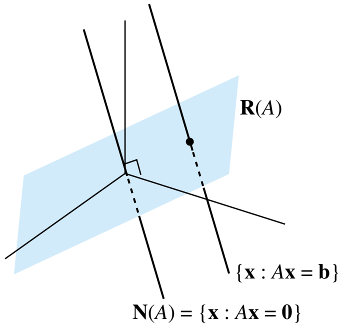


FIGURE 4.3

Remark. There is a further geometric interpretation of the vector $\mathbf{x}_0 \in \mathbf{R}(A)$ that arises in the preceding proposition. Of all the solutions of $A\mathbf{x} = \mathbf{b}$, the vector $\mathbf{x}_0$ is the one of least length. Why?

Exercises 3.4

1. Find a basis for each of the given subspaces and determine its dimension.
   *a. $V = \mathrm{Span}\big((1,2,3), (3,4,7), (5,-2,3)\big) \subset \mathbb{R}^3$
   b. $V = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0,\ x_2 + x_4 = 0\} \subset \mathbb{R}^4$
   c. $V = \big(\mathrm{Span}\big((1,2,3)\big)\big)^\perp \subset \mathbb{R}^3$
   d. $V = \{\mathbf{x} \in \mathbb{R}^5 : x_1 = x_2,\ x_3 = x_4\} \subset \mathbb{R}^5$

[6]These are, however, generally *not* inverse functions. Why? See Exercise 25.

2. In Example 3, we were given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} \in \mathbb{R}^4$ and constructed $\mathbf{v}_3, \mathbf{v}_4$ so that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ gives a basis for $\mathbb{R}^4$. Here is an alternative construction: Consider the collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$; these certainly span $\mathbb{R}^4$. Now use the approach of Example 2 to find a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$.

3. For each of the following matrices A, give bases for $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^{\mathsf{T}})$. Check dimensions and orthogonality.

a. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 3 & 3 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ -1 & 1 & -1 & 0 & -1 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & -1 & 1 \\ 2 & 2 & 2 & 0 & 0 \\ -1 & -1 & 2 & -3 & 3 \end{bmatrix}$

*f. $A = \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ -1 & 2 & 3 & 4 & 1 & -6 \\ 0 & 4 & 4 & 12 & -1 & -7 \end{bmatrix}$

4. Find a basis for the intersection of the subspaces

$$V = \text{Span}\big((1, 0, 1, 1), (2, 1, 1, 2)\big) \quad \text{and} \quad W = \text{Span}\big((0, 1, 1, 0), (2, 0, 1, 2)\big) \subset \mathbb{R}^4.$$

***5.** Give a basis for the orthogonal complement of each of the following subspaces of $\mathbb{R}^4$.

a. $V = \text{Span}\big((1, 0, 3, 4), (0, 1, 2, -5)\big)$

b. $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + 3x_3 + 4x_4 = 0,\ \ x_2 + 2x_3 - 5x_4 = 0\}$

6. a. Give a basis for the orthogonal complement of the subspace $V \subset \mathbb{R}^4$ given by

$$V = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 - 2x_4 = 0,\ x_1 - x_2 - x_3 + 6x_4 = 0,\ x_2 + x_3 - 4x_4 = 0\}.$$

b. Give a basis for the orthogonal complement of the subspace $W \subset \mathbb{R}^4$ spanned by $(1, 1, 0, -2)$, $(1, -1, -1, 6)$, and $(0, 1, 1, -4)$.

c. Give a matrix B so that the subspace W defined in part b can be written in the form $W = \mathbf{N}(B)$.

7. We saw in Exercise 2.5.13 that if the $m \times n$ matrix A has rank 1, then there are nonzero vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ such that $A = \mathbf{u}\mathbf{v}^{\mathsf{T}}$. Describe the four fundamental subspaces of A in terms of $\mathbf{u}$ and $\mathbf{v}$. (*Hint:* What are the columns of $\mathbf{u}\mathbf{v}^{\mathsf{T}}$?)

8. In each case, construct a matrix with the requisite properties or explain why no such matrix exists.

*a. The column space has basis $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and the nullspace contains $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

b. The nullspace contains $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and the row space contains $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

*c. The column space has basis $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$, and the row space has basis $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$.

d. The column space and the nullspace both have basis $\begin{bmatrix}1\\0\end{bmatrix}$.

e. The column space and the nullspace both have basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

9. Given the LU decompositions of the following matrices A, give bases for $\mathbf{R}(A)$, $\mathbf{N}(A)$, $\mathbf{C}(A)$, and $\mathbf{N}(A^{\mathsf{T}})$. (Do *not* multiply out!) Check dimensions and orthogonality.

a. $A = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 2 & 4\\0 & 1 & -1\end{bmatrix}$

*b. $A = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\-2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 2 & 1\\0 & 0 & 2 & -1\\0 & 0 & 0 & 0\end{bmatrix}$

c. $A = \begin{bmatrix}1 & 0 & 0 & 0\\-1 & 1 & 0 & 0\\0 & 2 & 1 & 0\\3 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & -2\\0 & 0 & 3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

10. According to Proposition 4.10, if A is an $m \times n$ matrix, then for each $\mathbf{b} \in \mathbf{C}(A)$, there is a unique $\mathbf{x} \in \mathbf{R}(A)$ with $A\mathbf{x} = \mathbf{b}$. In each case, give a formula for that $\mathbf{x}$.

a. $A = \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$

*b. $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & -1\end{bmatrix}$

11. Let $A = \begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & -1\end{bmatrix}$.

a. Given any $\mathbf{x} \in \mathbb{R}^4$, find $\mathbf{u} \in \mathbf{R}(A)$ and $\mathbf{v} \in \mathbf{N}(A)$ so that $\mathbf{x} = \mathbf{u} + \mathbf{v}$.

b. Given $\mathbf{b} \in \mathbb{R}^2$, give the unique element $\mathbf{x} \in \mathbf{R}(A)$ so that $A\mathbf{x} = \mathbf{b}$.

12. Let A be an $n \times n$ matrix. Prove that A is singular if and only if A^{T} is singular.

***13.** Let A be an $m \times n$ matrix with rank r. Suppose $A = BU$, where U is in echelon form. Show that the first r columns of B give a basis for $\mathbf{C}(A)$. (In particular, if $EA = U$, where U is the echelon form of A and E is the product of elementary matrices by which we reduce A to U, then the first r columns of E^{-1} give a basis for $\mathbf{C}(A)$.)

14. Recall from Exercise 3.1.13 that for any subspace $V \subset \mathbb{R}^n$ we have $V \subset (V^{\perp})^{\perp}$. Give alternative proofs of Proposition 3.6

*a. by applying Proposition 4.3;

b. by applying Theorem 4.9 to prove that if $\mathbf{x} \in (V^{\perp})^{\perp}$, then $\mathbf{x} \in V$.

15. Let $V \subset \mathbb{R}^n$ be a subspace, let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for V, and let $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in V$ be vectors such that Span $(\mathbf{w}_1, \dots, \mathbf{w}_\ell) = V$. Prove that $\ell \geq k$.

16. Prove Proposition 4.4. (*Hint:* Exercise 15 and Proposition 4.3 may be useful.)

♯**17.** Let $V \subset \mathbb{R}^n$ be a subspace, and suppose you are given a linearly independent set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$. Show that if $\dim V > k$, then there are vectors $\mathbf{v}_{k+1}, \dots, \mathbf{v}_\ell \in V$ so that $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ forms a basis for V.

18. Suppose V and W are subspaces of $\mathbb{R}^n$ and $W \subset V$. Prove that $\dim W \le \dim V$. (*Hint:* Start with a basis for W and apply Exercise 17.)

19. Suppose A is an $n \times n$ matrix, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$. Suppose $\{A\mathbf{v}_1, \dots, A\mathbf{v}_n\}$ is linearly independent. Prove that A is nonsingular.

20. Continuing Exercise 3.3.23: Let U and V be subspaces of $\mathbb{R}^n$. Prove that if $U \cap V = \{\mathbf{0}\}$, then $\dim(U + V) = \dim U + \dim V$.

21. Let U and V be subspaces of $\mathbb{R}^n$. Prove that $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$. (*Hint:* This is a generalization of Exercise 20. Start with a basis for $U \cap V$, and use Exercise 17.)

22. Continuing Exercise 3.2.10: Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix.
  a. Prove that $\text{rank}(AB) \le \text{rank}(A)$. (*Hint:* Look at part *b* of Exercise 3.2.10.)
  b. Prove that if $n = p$ and B is nonsingular, then $\text{rank}(AB) = \text{rank}(A)$.
  c. Prove that $\text{rank}(AB) \le \text{rank}(B)$. (*Hint:* Use part *a* of Exercise 3.2.10 and Theorem 4.6.)
  d. Prove that if $m = n$ and A is nonsingular, then $\text{rank}(AB) = \text{rank}(B)$.
  e. Prove that if $\text{rank}(AB) = n$, then $\text{rank}(A) = \text{rank}(B) = n$.

23. a. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Show that $AB = O \iff \mathbf{C}(B) \subset \mathbf{N}(A)$.
  b. Suppose A and B are 3×3 matrices of rank 2. Show that $AB \ne O$.
  c. Give examples of 3×3 matrices A and B of rank 2 so that AB has each possible rank.

♯**24.** Continuing Exercise 3.2.10: Let A be an $m \times n$ matrix.
  a. Use Theorem 2.5 to prove that $\mathbf{N}(A^\mathsf{T}A) = \mathbf{N}(A)$. (*Hint:* If $\mathbf{x} \in \mathbf{N}(A^\mathsf{T}A)$, then $A\mathbf{x} \in \mathbf{C}(A) \cap \mathbf{N}(A^\mathsf{T})$.)
  b. Prove that $\text{rank}(A) = \text{rank}(A^\mathsf{T}A)$.
  c. Prove that $\mathbf{C}(A^\mathsf{T}A) = \mathbf{C}(A^\mathsf{T})$.

25. In this exercise, we investigate the composition of functions $\mu_{A^\mathsf{T}} \circ \mu_A$ mapping $\mathbf{R}(A)$ to $\mathbf{R}(A)$, pursuing the discussion on p. 167.
  a. Suppose A is an $m \times n$ matrix. Show that $A^\mathsf{T}A = I_n$ if and only if the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$ are mutually orthogonal unit vectors.
  b. Suppose A is an $m \times n$ matrix of rank 1. Using the notation of Exercise 7, show that $A^\mathsf{T}A\mathbf{x} = \mathbf{x}$ for each $\mathbf{x} \in \mathbf{R}(A)$ if and only if $\|\mathbf{u}\|\|\mathbf{v}\| = 1$. Use this fact to show that we can write $A = \hat{\mathbf{u}}\hat{\mathbf{v}}^\mathsf{T}$, where $\|\hat{\mathbf{u}}\| = \|\hat{\mathbf{v}}\| = 1$. Interpret μ_A geometrically. (See Exercise 4.4.22 for a generalization.)

26. Generalizing Exercise 3.2.13: Suppose A is an $n \times n$ matrix with the property that $\text{rank}(A) = \text{rank}(A^2)$.
  a. Show that $\mathbf{N}(A^2) = \mathbf{N}(A)$.
  b. Prove that $\mathbf{C}(A) \cap \mathbf{N}(A) = \{\mathbf{0}\}$.
  c. Conclude that $\mathbf{C}(A) + \mathbf{N}(A) = \mathbb{R}^n$. (*Hint:* Use Exercise 20.)

5 A Graphic Example

In this brief section we show how the four fundamental subspaces (and their orthogonality relations) have a natural interpretation in elementary graph theory, a subject that has many applications in computer science, electronics, applied mathematics, and topology. We also reinterpret Kirchhoff's laws from Section 6.3 of Chapter 1.

A *directed graph* consists of finitely many *nodes* (vertices) and *directed edges*, where the direction is indicated by an arrow on the edge. Each edge begins at one node and ends at a different node; moreover, edges can meet only at nodes. Of course, given a directed graph, we can forget about the arrows and consider the associated *undirected* graph. A *path* in a graph is a (finite) sequence of nodes, $v_1, v_2, \dots, v_k$, along with the choice, for each $j = 1, \dots, k-1$, of *some* edge with endpoints at v_j and v_{j+1}. (Note that if we wish to make the path a *directed* path, then we must require, in addition, that the respective edges begin at v_j and end at v_{j+1}.) The path is called a *loop* if $v_1 = v_k$. An undirected graph is called *connected* if there is a path from any node to any other node. It is called *complete* if there is a single edge joining every pair of nodes.

Given a directed graph with m edges and n nodes, we start by numbering the edges 1 through m and the nodes 1 through n. Then there is an obvious $m \times n$ matrix for us to write down, called the *incidence matrix* of the graph. Define $A = \left[a_{ij}\right]$ by the rule

$$a_{ij} = \begin{cases} -1, & \text{if edge } i \text{ starts at node } j \\ +1, & \text{if edge } i \text{ ends at node } j \\ 0, & \text{otherwise.} \end{cases}$$

Notice that each row of this matrix contains exactly one $+1$ and exactly one -1 (and all other entries are 0), because each edge starts at precisely one node and ends at precisely one node. For the graphs in Figure 5.1

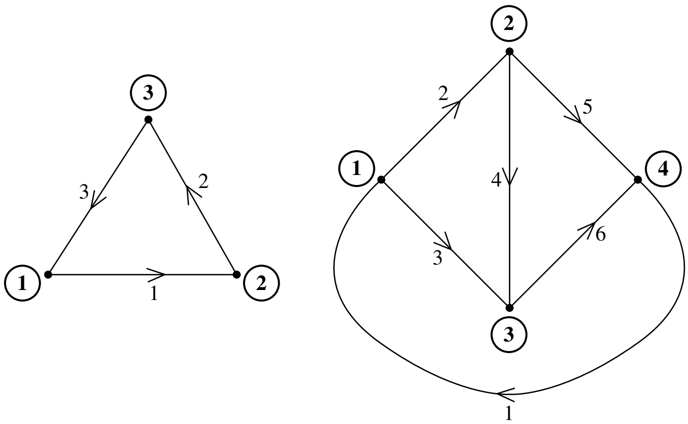


FIGURE 5.1

we have the respective incidence matrices

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

We offer a few possible interpretations of the vector equation $A\mathbf{x} = \mathbf{y}$. If $x_1, \dots, x_n$ are the electric potentials at the n nodes, then $y_1, \dots, y_m$ are the potential differences (voltages) across the m edges (wires); these result in current flow, as we shall soon see. If the nodes represent scenic sites along a mountain road, we can take $x_1, \dots, x_n$ to be their elevations, and then $y_1, \dots, y_m$ give the change in elevation along the edges (roadways). If the edges represent pipes and the nodes are joints, then we might let $x_1, \dots, x_n$ represent water pressure at the respective joints, and then $y_1, \dots, y_m$ will be the pressure differences across the pipes (which ultimately result in water flow).

The transpose matrix A^{T} also has a nice geometric interpretation. Given a path in the directed graph, define a vector $\mathbf{z} = (z_1, \dots, z_m)$ as follows: Let z_i be the net number of times that edge i is traversed as directed, i.e., the difference between the number of times it is traversed forward and the number of times it is traversed backward. Since the columns of A^{T} are the rows of A, the vector $A^{\mathsf{T}}\mathbf{z}$ gives a linear combination of the rows of A, each corresponding to an edge. Thus, $A^{\mathsf{T}}\mathbf{z}$ will be a vector whose coefficients indicate the starting and ending nodes of the path. For example, in the second graph in Figure 5.1, the path from node 1 to node 2 to node 4 to node 3 is represented by the vector

$$\mathbf{z} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad A^{\mathsf{T}}\mathbf{z} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

indicates that the path starts at node 1 and ends at node 3 (in other words, that the *boundary* of the path is "node 3 − node 1"). Note that the path represented by the vector $\mathbf{z}$ is a loop when $A^{\mathsf{T}}\mathbf{z} = \mathbf{0}$. For these reasons, A^{T} is called the *boundary matrix* associated to the graph. Thus, vectors in $\mathbf{N}(A^{\mathsf{T}})$ (at least those with integer entries) tell us the loops in the graph, or, equivalently, which linear combinations of the row vectors of A give $\mathbf{0}$. It is not hard to see that in the case of the first graph in Figure 5.1,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

gives a basis for $\mathbf{N}(A^{\mathsf{T}})$, as does

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

in the case of the second graph.

When we consider elements of $\mathbf{R}(A) = \mathbf{C}(A^{\mathsf{T}})$ with integer entries, we're asking for all of the possible vectors that arise as boundaries of paths (or as boundaries of "sums" of paths). For example, in the case of the first graph in Figure 5.1,

$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

is the boundary of the "path" that traverses edge 2 once and edge 3 twice, but

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

cannot be a boundary. The reason, intuitively, is that when we traverse an edge, its endpoints must "cancel" in sign, so when we form a path, the total number of $+1$'s and the total number of -1's must be equal. To give a more precise argument, we should use the result of Theorem 2.5 that $\mathbf{R}(A) = \mathbf{N}(A)^{\perp}$, to which end we examine $\mathbf{N}(A)$ next.

Now, $\mathbf{x} \in \mathbf{N}(A)$ if and only if $A\mathbf{x} = \mathbf{0}$, which tells us that the voltage (potential difference) across each edge is 0. In other words, if we specify the potential x_1 at the first node, then any node that is joined to the first node by an edge must have the same potential. This means that when the graph is *connected*—that is, when there is a path joining any pair of its nodes—the vector

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

must span $\mathbf{N}(A)$. (In general, $\dim \mathbf{N}(A)$ equals the number of connected "pieces.") More conceptually, suppose the graph is connected and we know the voltages in all the edges. Can we compute the potentials at the nodes? Only once we "ground" one of the nodes, i.e., specify that it has potential 0. (See Theorem 5.3 of Chapter 1. This is rather like adding the constant of integration in calculus. But we'll see why in the next chapter!)

Suppose now that the graph is connected. Since $\mathbf{R}(A)^{\perp}$ is spanned by the vector $\mathbf{a}$ above, we conclude that

$$\mathbf{w} \in \mathbf{R}(A) \iff w_1 + w_2 + \cdots + w_n = 0,$$

which agrees with our intuitive discussion earlier. Indeed, in general, we see that the same cancellation principle must hold in every connected piece of our graph, so there is one such constraint for every piece.

What is the column space of A? For what vectors $\mathbf{y}$ can we solve $A\mathbf{x} = \mathbf{y}$? That is, for what voltages (along the edges) can we arrange potentials (at the nodes) with the correct differences? We know from Theorem 2.5 that $\mathbf{C}(A) = \mathbf{N}(A^{\mathsf{T}})^{\perp}$, so $\mathbf{y} \in \mathbf{C}(A)$ if and only if $\mathbf{v} \cdot \mathbf{y} = 0$ for every $\mathbf{v} \in \mathbf{N}(A^{\mathsf{T}})$. In the case of our examples, this says that in the first example, $\mathbf{y}$ must satisfy the constraint equation

$$y_1 + y_2 + y_3 = 0;$$

and in the second example, $\mathbf{y}$ must satisfy the constraint equations

$$\begin{aligned} y_1 \quad\;\; + y_3 \qquad\qquad\quad + y_6 &= 0 \\ y_2 - y_3 + y_4 \qquad\qquad &= 0 \\ y_1 + y_2 \qquad\qquad + y_5 \qquad &= 0. \end{aligned}$$

Since elements of $\mathbf{N}(A^{\mathsf{T}})$ correspond to loops in the circuit, we recognize Kirchhoff's second law from Section 6.3 of Chapter 1: The net voltage drop around any loop in the circuit must be 0. We now know that this condition is sufficient to solve for the voltages in the wires.

To complete the discussion of electric circuits in our present setting, we need to introduce the vector of currents. Given a circuit with m edges (wires), let z_i denote the current in the i^{th} wire, and set $\mathbf{z} = (z_1, \ldots, z_m)$. Then Kirchhoff's first law, which states that the total current coming into a node equals the total current leaving the node, can be rephrased very simply as

$$A^{\mathsf{T}}\mathbf{z} = \mathbf{0}.$$

In the case of the first example, we have

$$\begin{aligned} -z_1 \qquad\quad + z_3 &= 0 \\ z_1 - z_2 \qquad\quad &= 0 \\ z_2 - z_3 &= 0, \end{aligned}$$

and there are four equations, correspondingly, in the second:

$$\begin{aligned} z_1 - z_2 - z_3 \qquad\qquad\qquad\qquad &= 0 \\ z_2 \qquad - z_4 - z_5 \qquad\quad &= 0 \\ z_3 + z_4 \qquad - z_6 &= 0 \\ -z_1 \qquad\qquad\qquad + z_5 + z_6 &= 0, \end{aligned}$$

just as we obtained earlier in Chapter 1.

If the reader is interested in pursuing the discussion of Ohm's law and Kirchhoff's second law further, we highly recommend Exercise 5.

Exercises 3.5

*1. Give the dimensions of the four fundamental subspaces of the incidence matrix A of the graph in Figure 5.2. Explain your answer in each case. Also compute $A^{\mathsf{T}}A$ and interpret its entries in terms of properties of the graph.

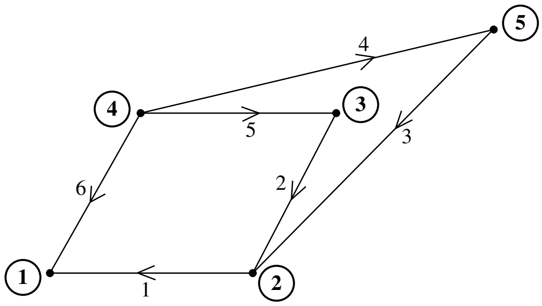


FIGURE 5.2

2. Let A denote the incidence matrix of the disconnected graph shown in Figure 5.3. Use the geometry of the graph to answer the following.
   a. Give a basis for $\mathbf{N}(A^{\mathsf{T}})$.
   b. Give a basis for $\mathbf{N}(A)$.
   c. Give the equations $\mathbf{y}$ must satisfy if $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$.

*3. Give the dimensions of the four fundamental subspaces of the incidence matrix A of the *complete* graph shown in Figure 5.4. Explain your answer geometrically in each case. (What happens in the case of a complete graph with n nodes?)

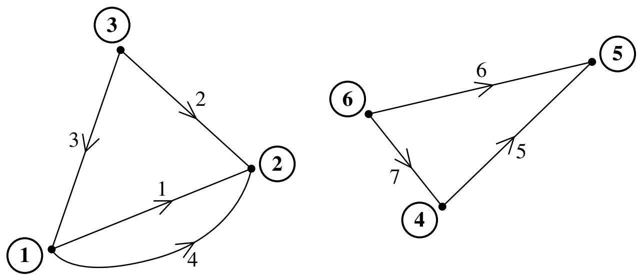


FIGURE 5.3

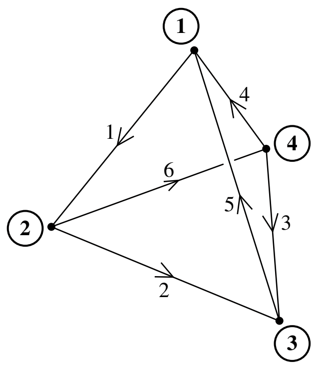


FIGURE 5.4

4. a. Show that in a graph with n nodes and n edges, there must be a loop.

b. A graph is called a *tree* if it contains no loops. Show that if a graph is a tree with n nodes, then it has at most $n-1$ edges. (Thus, a tree with n nodes and $n-1$ edges is called a *maximal tree*.)

c. Give an example of an incidence matrix A with $\dim \mathbf{N}(A) > 1$. Draw a picture of the graph corresponding to your matrix and explain.

5. Ohm's Law says that $V = IR$; that is, voltage (in volts) = current (in amps) × resistance (in ohms). Given an electric circuit with m wires, let R_i denote the resistance in the i^{th} wire, and let y_i and z_i denote, respectively, the voltage drop across and current in the i^{th} wire, as in the text. Let $\mathcal{E} = (E_1, \dots, E_m)$, where E_i is the external voltage source in the i^{th} wire, and let C be the diagonal $m \times m$ matrix whose ii-entry is R_i, $i = 1, \dots, m$; we assume that all $R_i > 0$. Then we have $\mathbf{y} + \mathcal{E} = C\mathbf{z}$. Let A denote the incidence matrix for this circuit.

a. Prove that for every $\mathbf{v} \in \mathbf{N}(A^{\mathsf{T}})$, we have $\mathbf{v} \cdot C\mathbf{z} = \mathbf{v} \cdot \mathcal{E}$, and compare this with the statement of Kirchhoff's second law in Section 6.3 of Chapter 1.

b. Assume the network is connected, so that $\operatorname{rank}(A) = n-1$; delete a column of A (say the last) and call the resulting matrix $\tilde{A}$. This amounts to grounding the last node. Generalize the result of Exercise 3.4.24 to prove that $\tilde{A}^{\mathsf{T}} C^{-1} \tilde{A}$ is nonsingular. (*Hint:* Write $C = D^2 = DD^{\mathsf{T}}$, where D is the diagonal matrix with entries $\sqrt{R_i}$.)

c. Deduce that for any external voltage sources $\mathcal{E}$, there is a unique solution of the equation $(\tilde{A}^{\mathsf{T}} C^{-1} \tilde{A})\mathbf{x} = \tilde{A}^{\mathsf{T}} C^{-1} \mathcal{E}$.

d. Deduce that for any external voltage sources $\mathcal{E}$, the currents in the network are uniquely determined.

6. Use the approach of Exercise 5 to obtain the answer to Example 5 in Section 6.3 of Chapter 1.

6 Abstract Vector Spaces

We have seen throughout this chapter that subspaces of $\mathbb{R}^n$ behave algebraically much the same way as $\mathbb{R}^n$ itself. They are endowed with two operations: vector addition and scalar multiplication. But there are lots of other interesting collections of objects—which do not in any obvious way live in some Euclidean space $\mathbb{R}^n$—that have the same algebraic properties. In order to study some of these collections and see some of their applications, we first make the following general definition.

Definition. A (real) *vector space* V is a set that is equipped with two operations, vector addition and scalar multiplication, which satisfy the following properties:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There is a vector $\mathbf{0} \in V$ (the zero vector) such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.
4. For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. For all $c, d \in \mathbb{R}$ and $\mathbf{u} \in V$, $c(d\mathbf{u}) = (cd)\mathbf{u}$.
6. For all $c \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
7. For all $c, d \in \mathbb{R}$ and $\mathbf{u} \in V$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
8. For all $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$.

EXAMPLE 1

(a) $\mathbb{R}^n$ is, of course, a vector space, as is any subspace of $\mathbb{R}^n$.

(b) The empty set is not a vector space, since it does not satisfy condition (3) of the definition. All the remaining criteria are satisfied "by default."

(c) Let $\mathcal{M}_{m\times n}$ denote the set of all $m \times n$ matrices. As we've seen in Proposition 1.1 of Chapter 2, $\mathcal{M}_{m\times n}$ is a vector space, using the operations of matrix addition and scalar multiplication we've already defined. The zero "vector" is the zero matrix O.

(d) Let $\mathcal{F}(\mathcal{I})$ denote the collection of all real-valued functions defined on some interval $\mathcal{I} \subset \mathbb{R}$. If $f \in \mathcal{F}(\mathcal{I})$ and $c \in \mathbb{R}$, then we can define a new function $cf \in \mathcal{F}(\mathcal{I})$ by multiplying the *value* of f at each point by the scalar c:

$$(cf)(t) = c\,f(t) \quad \text{for each } t \in \mathcal{I}.$$

Similarly, if $f, g \in \mathcal{F}(\mathcal{I})$, then we can define the new function $f + g \in \mathcal{F}(\mathcal{I})$ by adding the *values* of f and g at each point:

$$(f + g)(t) = f(t) + g(t) \quad \text{for each } t \in \mathcal{I}.$$

By these formulas we define scalar multiplication and vector addition in $\mathcal{F}(\mathcal{I})$. The zero "vector" in $\mathcal{F}(\mathcal{I})$ is the zero function, whose value at each point is 0:

$$\mathbf{0}(t) = 0 \quad \text{for each } t \in \mathcal{I}.$$

The various properties of a vector space follow from the corresponding properties of the real numbers (because everything is defined in terms of the *values* of the function at every point t). Since an element of $\mathcal{F}(\mathcal{I})$ is a function, $\mathcal{F}(\mathcal{I})$ is often called a *function space*.

(e) Let $\mathbb{R}^\omega$ denote the collection of all infinite sequences of real numbers. That is, an element of $\mathbb{R}^\omega$ looks like

$$\mathbf{x} = (x_1, x_2, x_3, \dots),$$

where all the x_i's are real numbers. Operations are defined in the obvious way: If $c \in \mathbb{R}$, then

$$c\mathbf{x} = (cx_1, cx_2, cx_3, \dots),$$

and if $\mathbf{y} = (y_1, y_2, y_3, \dots) \in \mathbb{R}^\omega$, we define addition by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots).$$ ▲

As we did with $\mathbb{R}^n$, we can define subspaces of a general vector space, and they too can be considered as vector spaces in their own right.

Definition. Let V be a vector space. We say $W \subset V$ is a *subspace* if

1. $\mathbf{0} \in W$ (the zero vector belongs to W);
2. whenever $\mathbf{v} \in W$ and $c \in \mathbb{R}$, we have $c\mathbf{v} \in W$ (W is closed under scalar multiplication);
3. whenever $\mathbf{v}, \mathbf{w} \in W$, we have $\mathbf{v} + \mathbf{w} \in W$ (W is closed under addition).

Proposition 6.1. *If V is a vector space and $W \subset V$ is a subspace, then W is also a vector space.*

Proof. We need to check that W satisfies the eight properties in the definition. The crucial point is that the definition of subspace ensures that when we perform vector addition or scalar multiplication starting with vector(s) in W, the result again lies in W. All the algebraic properties hold in W because they already hold in V. Only one subtle point should be checked: Given $\mathbf{w} \in W$, we know $\mathbf{w}$ has an additive inverse $-\mathbf{w}$ in V, but why must this vector lie in W? We leave this to the reader to check (but, for a hint, see part b of Exercise 1). □

EXAMPLE 2

Consider the vector space $\mathcal{M}_{n\times n}$ of square matrices. Then the following are subspaces:

$$\mathcal{U} = \{\text{upper triangular matrices}\}$$
$$\mathcal{D} = \{\text{diagonal matrices}\}$$
$$\mathcal{L} = \{\text{lower triangular matrices}\}.$$

We ask the reader to check the details in Exercise 5. ▲

EXAMPLE 3

Fix an $n \times n$ matrix M and let $W = \{A \in \mathcal{M}_{n\times n} : AM = MA\}$. This is the set of matrices that commute with M. We ask whether W is a subspace of $\mathcal{M}_{n\times n}$. Since $\mathrm{O}M = \mathrm{O} = M\mathrm{O}$, it follows that $\mathrm{O} \in W$. If c is a scalar and $A \in W$, then $(cA)M = c(AM) = c(MA) = M(cA)$, so $cA \in W$ as well. Last, suppose $A, B \in W$. Then $(A + B)M = AM + BM = MA + MB = M(A + B)$, so $A + B \in W$, as we needed. This completes the verification that W is a subspace of $\mathcal{M}_{n\times n}$. ▲

EXAMPLE 4

There are many interesting—and extremely important—subspaces of $\mathcal{F}(\mathcal{I})$. The space $\mathcal{C}^0(\mathcal{I})$ of *continuous* functions on $\mathcal{I}$ is a subspace because of the important result from calculus that states: If f is a continuous function, then so is cf for any $c \in \mathbb{R}$; and if f and g are continuous functions, then so is $f + g$. In addition, the zero function is continuous.

The space $\mathcal{D}(\mathcal{I})$ of *differentiable* functions is a subspace of $\mathcal{C}^0(\mathcal{I})$. First, every differentiable function is continuous. Next, we have an analogous result from calculus that states: If f is a differentiable function, then so is cf for any $c \in \mathbb{R}$; and if f and g are differentiable functions, then so is $f + g$. Likewise, the zero function is differentiable. Indeed, we know more from our calculus class. We have formulas for the relevant derivatives:

$$(cf)' = cf' \quad \text{and} \quad (f + g)' = f' + g',$$

which will be important in Chapter 4.

Mathematicians tend to concentrate on the subspace $\mathcal{C}^1(\mathcal{I}) \subset \mathcal{D}(\mathcal{I})$ of *continuously differentiable* functions. (A function f is continuously differentiable if its derivative f' is a continuous function.) They then move down the hierarchy:

$$\mathcal{C}^0(\mathcal{I}) \supset \mathcal{C}^1(\mathcal{I}) \supset \mathcal{C}^2(\mathcal{I}) \supset \mathcal{C}^3(\mathcal{I}) \supset \cdots \supset \mathcal{C}^k(\mathcal{I}) \supset \mathcal{C}^{k+1}(\mathcal{I}) \supset \cdots \supset \mathcal{C}^\infty(\mathcal{I}),$$

where $\mathcal{C}^k(\mathcal{I})$ is the collection of functions on $\mathcal{I}$ that are (at least) k times continuously differentiable and $\mathcal{C}^\infty(\mathcal{I})$ is the collection of functions on $\mathcal{I}$ that are infinitely differentiable. The reader who's had some experience with mathematical induction can easily prove these are all subspaces. ▲

EXAMPLE 5

Let $W \subset \mathbb{R}^\omega$ denote the set of all sequences that are eventually 0. That is, let

$$W = \{\mathbf{x} \in \mathbb{R}^\omega : \text{there is a positive integer } n \text{ such that } x_k = 0 \text{ for all } k > n\}.$$

Then we claim that W is a subspace of $\mathbb{R}^\omega$.

Clearly, $\mathbf{0} \in W$. If $\mathbf{x} \in W$, then there is an integer n such that $x_k = 0$ for all $k > n$, so, for any scalar c, we know that $cx_k = 0$ for all $k > n$. Therefore, the k^{th} coordinate of $c\mathbf{x}$ is 0 for all $k > n$ and so $c\mathbf{x} \in W$ as well. Now, suppose $\mathbf{x}$ and $\mathbf{y} \in W$. Then there are integers n and p such that $x_k = 0$ for all $k > n$ and $y_k = 0$ for all $k > p$. It follows that $x_k + y_k = 0$ for all k larger than both n and p. (Officially, we write this as $k > \max(n, p)$.) Therefore, $\mathbf{x} + \mathbf{y} \in W$. ▲

Of particular importance to us here is the vector space $\mathcal{P}$ of polynomials. Recall that a polynomial p of degree k is a function of the form

$$p(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0,$$

where $a_0, a_1, \ldots, a_k$ are real numbers and $a_k \neq 0$. If $c \in \mathbb{R}$, then the scalar multiple cp is given by the formula

$$(cp)(t) = ca_k t^k + ca_{k-1} t^{k-1} + \cdots + ca_1 t + ca_0.$$

If q is another polynomial of degree k, say

$$q(t) = b_k t^k + b_{k-1} t^{k-1} + \cdots + b_1 t + b_0,$$

then the sum $p + q$ is again a polynomial:

$$(p + q)(t) = (a_k + b_k) t^k + (a_{k-1} + b_{k-1}) t^{k-1} + \cdots + (a_1 + b_1) t + (a_0 + b_0).$$

But be careful! The sum of two polynomials of degree k may well have lesser degree. (For example, let $p(t) = t^2 - 2t + 3$ and $q(t) = -t^2 + t - 1$.) Thus, it is also natural to consider the subspaces $\mathcal{P}_k$ of polynomials of degree *at most* k, including the zero polynomial. We will return to study the spaces $\mathcal{P}_k$ shortly.

It is easy to extend the notions of span, linear independence, basis, and dimension to the setting of general vector spaces. We briefly restate the definitions here to be sure.

Definition. Let V be a vector space and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$. Then the *span* of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the set

$$\text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \dots, c_k \in \mathbb{R}\}.$$

We say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \quad \text{only when} \quad c_1 = c_2 = \cdots = c_k = 0.$$

Otherwise, we say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is *linearly dependent*.

We say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a *basis* for V if

(i) $\mathbf{v}_1, \dots, \mathbf{v}_k$ span V, i.e., $V = \text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$, and

(ii) $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

EXAMPLE 6

Consider the polynomials $p_1(t) = t + 1$, $p_2(t) = t^2 + 2$, and $p_3(t) = t^2 - t$. We want to decide whether $\{p_1, p_2, p_3\} \subset \mathcal{P}$ is a linearly independent set of vectors. Suppose $c_1 p_1 + c_2 p_2 + c_3 p_3 = \mathbf{0}$. This means

$$c_1(t + 1) + c_2(t^2 + 2) + c_3(t^2 - t) = 0 \quad \text{for all } t.$$

By specifying different values of t, we may obtain a homogeneous system of linear equations in the variables c_1, c_2, and c_3:

$$\begin{aligned} t = -1: &\qquad\quad 3c_2 + 2c_3 = 0 \\ t = 0: &\quad c_1 + 2c_2 \qquad\quad = 0 \\ t = 1: &\quad 2c_1 + 3c_2 \qquad\quad = 0. \end{aligned}$$

We leave it to the reader to check that the only solution is $c_1 = c_2 = c_3 = 0$, and so the functions p_1, p_2, and p_3 do indeed form a linearly independent set. ▲

At this point, we stop to make one new definition.

Definition. Let V be a vector space. We say V is *finite-dimensional* if there are an integer k and vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ that form a basis for V. A vector space that is not finite-dimensional is called *infinite-dimensional*.

EXAMPLE 7

Observe that for any positive integer n, $\mathbb{R}^n$ is naturally a subspace of $\mathbb{R}^\omega$: Given a vector $(x_1, \dots, x_n) \in \mathbb{R}^n$, merely consider the corresponding vector $(x_1, \dots, x_n, 0, \dots) \in \mathbb{R}^\omega$. Since $\mathbb{R}^n$ is a subspace of $\mathbb{R}^\omega$ for every positive integer n, $\mathbb{R}^\omega$ contains an n-dimensional subspace for every positive integer n. It follows that $\mathbb{R}^{\omega}$ is infinite-dimensional. What about the subspace W of sequences that are eventually 0, defined in Example 5? Well, the same argument is valid, since every $\mathbb{R}^n$ is also a subspace of W. ▲

The reader should check that all the results of Sections 1, 2, 3, and 4 that applied to subspaces of $\mathbb{R}^n$ apply to any finite-dimensional vector space. The one argument that truly relied on coordinates was the proof of Proposition 4.1, which allowed us to conclude that dimension is well-defined. We now give a proof that works in general.

Proposition 6.2. *Let V be a finite-dimensional vector space and suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for V. If $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in V$ and $\ell > k$, then $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ must be linearly dependent.*

Proof. As in the proof of Proposition 4.1, we write

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{k1}\mathbf{v}_k \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{k2}\mathbf{v}_k \\
&\ \ \vdots \\
\mathbf{w}_\ell &= a_{1\ell}\mathbf{v}_1 + a_{2\ell}\mathbf{v}_2 + \cdots + a_{k\ell}\mathbf{v}_k
\end{aligned}$$

and form the $k \times \ell$ matrix $A = \big[a_{ij}\big]$. As before, since $\ell > k$, there is a nonzero vector $\mathbf{c} \in \mathbf{N}(A)$, and

$$\sum_{j=1}^{\ell} c_j \mathbf{w}_j = \sum_{j=1}^{\ell} c_j \left(\sum_{i=1}^{k} a_{ij}\mathbf{v}_i\right) = \sum_{i=1}^{k}\left(\sum_{j=1}^{\ell} a_{ij}c_j\right)\mathbf{v}_i = \mathbf{0}.$$

Consequently, there is a nontrivial relation among $\mathbf{w}_1, \dots, \mathbf{w}_\ell$, as we were to show. □

Now Theorem 4.2 follows just as before, and we see that the notion of dimension makes sense in arbitrary vector spaces. We will see in a moment that we've already encountered several infinite-dimensional vector spaces.

We next consider the dimension of $\mathcal{P}_k$, the vector space of polynomials of degree at most k. Let $f_0(t) = 1$, $f_1(t) = t$, $f_2(t) = t^2, \dots, f_k(t) = t^k$.

Proposition 6.3. *The set $\{f_0, f_1, \dots, f_k\}$ is a basis for $\mathcal{P}_k$. Thus, $\mathcal{P}_k$ is a $(k+1)$-dimensional vector space.*

Proof. We first check that $f_0, \dots, f_k$ span $\mathcal{P}_k$. Suppose $p \in \mathcal{P}_k$; then for appropriate $a_0, a_1, \dots, a_k \in \mathbb{R}$, we have

$$\begin{aligned}
p(t) &= a_k t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0 \\
&= a_k f_k(t) + a_{k-1} f_{k-1}(t) + \cdots + a_1 f_1(t) + a_0 f_0(t),
\end{aligned}$$

so

$$p = a_k f_k + a_{k-1} f_{k-1} + \cdots + a_1 f_1 + a_0 f_0$$

is a linear combination of $f_0, f_1, \dots, f_k$, as required.

How do we see that the functions $f_0, f_1, \dots, f_k$ form a linearly independent set? Suppose $c_0 f_0 + c_1 f_1 + \cdots + c_k f_k = \mathbf{0}$. This means that

$$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_k t^k = 0 \quad \text{for every } t \in \mathbb{R}.$$

Now, it is a fact from basic algebra (see Exercise 11) that a polynomial of degree $\le k$ can have at most k roots, unless it is the zero polynomial. Since $p(t) = 0$ for all $t \in \mathbb{R}$, p must be the zero polynomial; i.e., $c_0 = c_1 = \cdots = c_k = 0$, as required. □

Here is an example that indicates an alternative proof of the linear independence.

EXAMPLE 8

We will show directly that $\{f_0, f_1, f_2, f_3\}$ is linearly independent. Suppose $c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3 = \mathbf{0}$. We must show that $c_0 = c_1 = c_2 = c_3 = 0$. Let

$$p(t) = c_0 f_0(t) + c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3.$$

Since $p(t) = 0$ for all t, it follows that $p(0) = 0$, so $c_0 = 0$. Since polynomials are (infinitely) differentiable, and since the derivative of the zero function is the zero function, we obtain

$$p'(t) = c_1 + 2c_2 t + 3c_3 t^2 = 0 \quad \text{for all } t.$$

Once again, we have $p'(0) = 0$, and so $c_1 = 0$. Similarly,

$$p''(t) = 2c_2 + 6c_3 t = 0 \quad \text{for all } t \qquad \text{and} \qquad p'''(t) = 6c_3 = 0 \quad \text{for all } t,$$

and so, evaluating at 0 once again, we have $c_2 = c_3 = 0$. We conclude that $c_0 = c_1 = c_2 = c_3 = 0$, so $\{f_0, f_1, f_2, f_3\}$ is linearly independent. ▲

Since $\dim \mathcal{P}_k = k + 1$ for each nonnegative integer k, we see from the relations

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots \subsetneq \mathcal{P}_k \subsetneq \cdots \subsetneq \mathcal{P} \subsetneq \mathcal{C}^\infty(\mathbb{R}) \subsetneq \cdots \subsetneq \mathcal{C}^2(\mathbb{R}) \subsetneq \mathcal{C}^1(\mathbb{R}) \subsetneq \mathcal{C}^0(\mathbb{R}),$$

that $\mathcal{P}$, $\mathcal{C}^\infty(\mathbb{R})$, and $\mathcal{C}^k(\mathbb{R})$ all contain subspaces of arbitrarily large dimension. This means that $\mathcal{P}$ must be infinite-dimensional, and hence so must $\mathcal{C}^\infty(\mathbb{R})$ and $\mathcal{C}^k(\mathbb{R})$ for every $k \geq 0$.

EXAMPLE 9

Let $V = \{f \in \mathcal{C}^1(\mathbb{R}) : f'(t) = f(t) \text{ for all } t \in \mathbb{R}\}$. V is a subspace of $\mathcal{C}^1(\mathbb{R})$, since

1. the zero function clearly has this property;
2. if $f \in V$ and $c \in \mathbb{R}$, then $(cf)'(t) = cf'(t) = cf(t) = (cf)(t)$, so $cf \in V$;
3. if $f, g \in V$, then $(f + g)'(t) = f'(t) + g'(t) = f(t) + g(t) = (f + g)(t)$, so $f + g \in V$.

A rather obvious element of V is the function $f_1(t) = e^t$. We claim that f_1 spans V and hence provides a basis for V. To see this, let f be an arbitrary element of V and consider the function $g(t) = f(t)e^{-t}$. Then, differentiating, we have

$$g'(t) = f'(t)e^{-t} - f(t)e^{-t} = (f'(t) - f(t))e^{-t} = 0 \quad \text{for all } t \in \mathbb{R},$$

and so, by the Mean Value Theorem in differential calculus, $g(t) = c$ for some $c \in \mathbb{R}$. Thus, $f(t)e^{-t} = c$ and so $f(t) = ce^t$, as required. ▲

We will explore the relation between linear algebra and differential equations further in Section 3 of Chapter 7.

We have not yet mentioned the dot product in the setting of abstract vector spaces.

Definition. Let V be a real vector space. We say V is an *inner product space* if for every pair of elements $\mathbf{u}, \mathbf{v} \in V$ there is a real number $\langle \mathbf{u}, \mathbf{v} \rangle$, called the *inner product of* $\mathbf{u}$ *and* $\mathbf{v}$, such that:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$ and scalars c;
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all $\mathbf{v} \in V$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ only if $\mathbf{v} = \mathbf{0}$.

EXAMPLE 10

(a) Our usual dot product on $\mathbb{R}^n$ is, of course, an inner product.

(b) Fix $k+1$ distinct real numbers $t_1, t_2, \dots, t_{k+1}$ and define an inner product on $\mathcal{P}_k$ by the formula

$$\langle p, q \rangle = \sum_{i=1}^{k+1} p(t_i)q(t_i), \qquad p, q \in \mathcal{P}_k.$$

All the properties of an inner product are obvious except for the very last. If $\langle p, p \rangle = 0$, then $\sum_{i=1}^{k+1} p(t_i)^2 = 0$, and so we must have $p(t_1) = p(t_2) = \dots = p(t_{k+1}) = 0$. But, as we observed in the proof of Proposition 6.3, if a polynomial of degree $\leq k$ has (at least) $k+1$ roots, then it must be the zero polynomial (see Exercise 11).

(c) Let $\mathcal{C}^0([a,b])$ denote the vector space of continuous functions on the interval $[a,b]$. If $f, g \in \mathcal{C}^0([a,b])$, define

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt.$$

We verify that the defining properties hold.

1. $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f \rangle$.
2. $\langle cf, g \rangle = \int_a^b (cf)(t)g(t)\,dt = \int_a^b cf(t)g(t)\,dt = c\int_a^b f(t)g(t)\,dt = c\langle f, g \rangle$.
3. $$\begin{aligned}\langle f+g, h \rangle &= \int_a^b (f+g)(t)h(t)\,dt = \int_a^b \big(f(t)+g(t)\big)h(t)\,dt \\ &= \int_a^b \big(f(t)h(t)+g(t)h(t)\big)\,dt = \int_a^b f(t)h(t)\,dt + \int_a^b g(t)h(t)\,dt \\ &= \langle f, h \rangle + \langle g, h \rangle.\end{aligned}$$
4. $\langle f, f \rangle = \int_a^b f(t)^2\,dt \geq 0$ since $f(t)^2 \geq 0$ for all t. On the other hand, if $\langle f, f \rangle = \int_a^b \big(f(t)\big)^2\,dt = 0$, then since f is continuous and $f^2 \geq 0$, it must be the case that $f = 0$. (If not, we would have $f(t_0) \neq 0$ for some t_0, and then $f(t)^2$ would be positive on some small interval containing t_0; it would then follow that $\int_a^b f(t)^2\,dt > 0$.)

The same inner product can be defined on subspaces of $\mathcal{C}^0([a,b])$, e.g., $\mathcal{P}_k$.

(d) We define an inner product on $\mathcal{M}_{n\times n}$ in Exercise 9. ▲

If V is an inner product space, we define length, orthogonality, and the angle between vectors just as we did in $\mathbb{R}^n$. If $\mathbf{v} \in V$, we define its length to be $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. We say $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. Since the Cauchy-Schwarz Inequality can be established in general by following the proof of Proposition 2.3 of Chapter 1 *verbatim*, we can define the angle θ between $\mathbf{v}$ and $\mathbf{w}$ by the equation

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}.$$

And we can define orthogonal subspaces and orthogonal complements analogously.

EXAMPLE 11

Let $V = \mathcal{C}^0([-1,1])$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt$. Let $U \subset V$ be the subset of even functions, and let $W \subset V$ be the subset of odd functions. That is,

$$\begin{aligned}U &= \{f \in V : f(-t) = f(t) \quad \text{for all } t \in [-1,1]\}\\ W &= \{f \in V : f(-t) = -f(t) \quad \text{for all } t \in [-1,1]\}.\end{aligned}$$

We claim first that U and W are orthogonal subspaces of V. For suppose $g \in U$ and $h \in W$. Then

$$\langle g, h\rangle = \int_{-1}^{1} g(t)h(t)\,dt = \int_{-1}^{0} g(t)h(t)\,dt + \int_{0}^{1} g(t)h(t)\,dt$$

(making the change of variables $s = -t$ in the first integral)

$$= \int_{0}^{1} g(-s)h(-s)\,ds + \int_{0}^{1} g(t)h(t)\,dt$$
$$= -\int_{0}^{1} g(s)h(s)\,ds + \int_{0}^{1} g(t)h(t)\,dt = 0,$$

as required.

Now, more is true. Every function can be written as the sum of an even and an odd function, to wit,

$$f(t) = \underbrace{\tfrac{1}{2}\big(f(t) + f(-t)\big)}_{\text{even}} + \underbrace{\tfrac{1}{2}\big(f(t) - f(-t)\big)}_{\text{odd}}.$$

Therefore, we have $V = U + W$. We can now infer that $W = U^{\perp}$ and $U = W^{\perp}$. We just check the former. We've already established that $W \subset U^{\perp}$, so it remains only to show that if $f \in U^{\perp}$, then $f \in W$. Write $f = f_1 + f_2$, where $f_1 \in U$ and $f_2 \in W$. Then we have

$$0 = \langle f, f_1\rangle = \langle f_1 + f_2, f_1\rangle = \langle f_1, f_1\rangle + \langle f_2, f_1\rangle = \langle f_1, f_1\rangle,$$

since we've already shown that even and odd functions are orthogonal. Thus, $f_1 = 0$ and $f \in W$, as we needed to show. (This means that $(U^{\perp})^{\perp} = U$ in this instance.[7] See Exercise 21 for an infinite-dimensional example in which this equality fails.) ▲

We can use the inner product defined in Example 10(b) to prove the following important result about curve fitting (see the discussion in Section 6.1 of Chapter 1).

Theorem 6.4 (Lagrange Interpolation Formula). *Given $k+1$ points*

$$(t_1, b_1),\ (t_2, b_2), \dots,\ (t_{k+1}, b_{k+1})$$

in the plane with $t_1, t_2, \dots, t_{k+1}$ distinct, there is exactly one polynomial $p \in \mathcal{P}_k$ whose graph passes through the points.

Proof. We begin by explicitly constructing a basis for $\mathcal{P}_k$ consisting of mutually orthogonal vectors of length 1 with respect to the inner product defined in Example 10(b). That is, to start, we seek a polynomial $p_1 \in \mathcal{P}_k$ so that

$$p_1(t_1) = 1, \quad p_1(t_2) = 0, \quad \dots, \quad p_1(t_{k+1}) = 0.$$

The polynomial $q_1(t) = (t - t_2)(t - t_3)\cdots(t - t_{k+1})$ has the property that $q_1(t_j) = 0$ for $j = 2, 3, \dots, k+1$, and $q_1(t_1) = (t_1 - t_2)(t_1 - t_3)\cdots(t_1 - t_{k+1}) \neq 0$ (why?). So now we set

$$p_1(t) = \frac{(t - t_2)(t - t_3)\cdots(t - t_{k+1})}{(t_1 - t_2)(t_1 - t_3)\cdots(t_1 - t_{k+1})};$$

then, as desired, $p_1(t_1) = 1$ and $p_1(t_j) = 0$ for $j = 2, 3, \dots, k+1$. Similarly, we can define

$$p_2(t) = \frac{(t - t_1)(t - t_3)\cdots(t - t_{k+1})}{(t_2 - t_1)(t_2 - t_3)\cdots(t_2 - t_{k+1})}$$

[7] This proof is identical to that of Proposition 3.6, and it will work whenever there are subspaces U and W with the property that $U + W = V$ and $U \cap W = \{\mathbf{0}\}$.

and polynomials $p_3, \dots, p_{k+1}$ so that

$$p_i(t_j) = \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}.$$

Like the standard basis vectors in Euclidean space, $p_1, p_2, \dots, p_{k+1}$ are unit vectors in $\mathcal{P}_k$ that are orthogonal to one another. It follows from Exercise 3.3.10 that these vectors form a linearly independent set and hence a basis for $\mathcal{P}_k$ (why?). In Figure 6.1 we give the graphs of the Lagrange "basis polynomials" p_1, p_2, p_3 for $\mathcal{P}_2$ when $t_1 = -1$, $t_2 = 0$, and $t_3 = 2$.

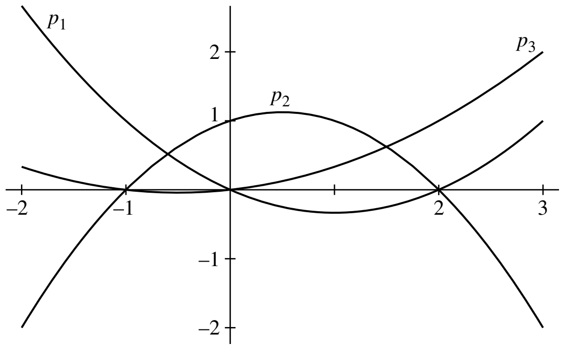


FIGURE 6.1

Now it is easy to see that the appropriate linear combination

$$p = b_1 p_1 + b_2 p_2 + \dots + b_{k+1} p_{k+1}$$

has the desired properties: $p(t_j) = b_j$ for $j = 1, 2, \dots, k+1$. On the other hand, two polynomials of degree $\leq k$ with the same values at $k+1$ points must be equal since their difference is a polynomial of degree $\leq k$ with at least $k+1$ roots. This establishes uniqueness. (More elegantly, any polynomial q with $q(t_j) = b_j$, $j = 1, \dots, k+1$, must satisfy $\langle q, p_j \rangle = b_j$, $j = 1, \dots, k+1$.) □

Remark. The proof worked so nicely because we constructed a basis that was adapted to the problem at hand. If we were to work with the "standard basis" $\{f_0, f_1, \dots, f_k\}$ for $\mathcal{P}_k$, we would need to find coefficients $a_0, \dots, a_k$ so that $p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k$ has the requisite properties. This results in a system of $k+1$ linear equations in the $k+1$ variables $a_0, \dots, a_k$:

$$\begin{aligned}
a_0 + a_1 t_1 + a_2 t_1^2 + \dots + a_k t_1^k &= b_1 \\
a_0 + a_1 t_2 + a_2 t_2^2 + \dots + a_k t_2^k &= b_2 \\
&\ \vdots \\
a_0 + a_1 t_{k+1} + a_2 t_{k+1}^2 + \dots + a_k t_{k+1}^k &= b_{k+1},
\end{aligned}$$

which in matrix form is

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^k \\ 1 & t_2 & t_2^2 & \dots & t_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{k+1} & t_{k+1}^2 & \dots & t_{k+1}^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k+1} \end{bmatrix}.$$

Theorem 6.4 shows that this system of equations has a unique solution so long as the t_i's are distinct. By Proposition 5.5 of Chapter 1, we deduce that the coefficient matrix

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^k \\ 1 & t_2 & t_2^2 & \dots & t_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{k+1} & t_{k+1}^2 & \dots & t_{k+1}^k \end{bmatrix}$$

is nonsingular, a fact that is amazingly tricky to prove by brute-force calculation (see Exercise 12 and also Exercise 4.4.20).

Exercises 3.6

1. Use the definition of a vector space V to prove the following:
   a. $0\mathbf{u} = \mathbf{0}$ for every $\mathbf{u} \in V$.
   b. $-\mathbf{u} = (-1)\mathbf{u}$ for every $\mathbf{u} \in V$.
   (*Hint:* The distributive property **7** is all important.)

*__2.__ Decide whether the following sets of vectors are linearly independent.
   a. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\} \subset \mathcal{M}_{2\times 2}$
   b. $\{f_1, f_2, f_3\} \subset \mathcal{P}_1$, where $f_1(t) = t,\ f_2(t) = t+1,\ f_3(t) = t+2$
   c. $\{f_1, f_2, f_3\} \subset \mathcal{C}^\infty(\mathbb{R})$, where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$
   d. $\{f_1, f_2, f_3\} \subset \mathcal{C}^0(\mathbb{R})$, where $f_1(t) = 1,\ f_2(t) = \sin^2 t,\ f_3(t) = \cos^2 t$
   e. $\{f_1, f_2, f_3\} \subset \mathcal{C}^\infty(\mathbb{R})$, where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \cos 2t$
   f. $\{f_1, f_2, f_3\} \subset \mathcal{C}^\infty(\mathbb{R})$, where $f_1(t) = 1,\ f_2(t) = \cos 2t,\ f_3(t) = \cos^2 t$

*__3.__ Decide whether each of the following is a subspace of $\mathcal{M}_{2\times 2}$. If so, provide a basis. If not, give a reason.
   a. $\left\{ A \in \mathcal{M}_{2\times 2} : \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbf{N}(A) \right\}$
   b. $\left\{ A \in \mathcal{M}_{2\times 2} : \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbf{C}(A) \right\}$
   c. $\{A \in \mathcal{M}_{2\times 2} : \operatorname{rank}(A) = 1\}$
   d. $\{A \in \mathcal{M}_{2\times 2} : \operatorname{rank}(A) \le 1\}$
   e. $\{A \in \mathcal{M}_{2\times 2} : A \text{ is in echelon form}\}$
   f. $\left\{ A \in \mathcal{M}_{2\times 2} : A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A \right\}$
   g. $\left\{ A \in \mathcal{M}_{2\times 2} : A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A^{\mathsf{T}} \right\}$

*__4.__ What is the dimension of the vector space $\mathcal{M}_{m\times n}$? Give a basis.

*__5.__ Check that the subsets defined in Example 2 are in fact subspaces of $\mathcal{M}_{n\times n}$. Find their dimensions.

6. Decide whether each of the following is a subspace of $\mathcal{C}^0(\mathbb{R})$. If so, provide a basis and determine its dimension. If not, give a reason.

*a. $\{f : f(1) = 2\}$

b. $\{f \in \mathcal{P}_2 : \int_0^1 f(t)\,dt = 0\}$

*c. $\{f \in \mathcal{C}^1(\mathbb{R}) : f'(t) + 2f(t) = 0 \text{ for all } t\}$

d. $\{f \in \mathcal{P}_4 : f(t) - tf'(t) = 0 \text{ for all } t\}$

e. $\{f \in \mathcal{P}_4 : f(t) - tf'(t) = 1 \text{ for all } t\}$

*f. $\{f \in \mathcal{C}^2(\mathbb{R}) : f''(t) + f(t) = 0 \text{ for all } t\}$

*g. $\{f \in \mathcal{C}^2(\mathbb{R}) : f''(t) - f'(t) - 6f(t) = 0 \text{ for all } t\}$

h. $\{f \in \mathcal{C}^1(\mathbb{R}) : f(t) = \int_0^t f(s)\,ds \text{ for all } t\}$

7. Suppose $a_0(t), \dots, a_{n-1}(t)$ are continuous functions. Prove that the set of solutions $y(t)$ of the n^{th}-order differential equation

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

is a subspace of $\mathcal{C}^n(\mathbb{R})$. Here $y^{(k)}(t)$ denotes the k^{th} derivative of the function $y(t)$. (See Theorem 3.4 of Chapter 7 for an algorithm for finding those solutions when the functions $a_j(t)$ are constants.)

8. Let $\mathcal{M}_{n\times n}$ denote the vector space of all $n \times n$ matrices.

a. Let $\mathcal{S} \subset \mathcal{M}_{n\times n}$ denote the set of symmetric matrices (those satisfying $A^{\mathsf{T}} = A$). Show that $\mathcal{S}$ is a subspace of $\mathcal{M}_{n\times n}$. What is its dimension?

b. Let $\mathcal{K} \subset \mathcal{M}_{n\times n}$ denote the set of skew-symmetric matrices (those satisfying $A^{\mathsf{T}} = -A$). Show that $\mathcal{K}$ is a subspace of $\mathcal{M}_{n\times n}$. What is its dimension?

c. Show that $\mathcal{S} + \mathcal{K} = \mathcal{M}_{n\times n}$. (See Exercise 2.5.22.)

9. Define the *trace* of an $n \times n$ matrix A (denoted $\text{tr}A$) to be the sum of its diagonal entries:

$$\text{tr}A = \sum_{i=1}^{n} a_{ii}.$$

a. Show that $\text{tr}A = \text{tr}(A^{\mathsf{T}})$.

b. Show that $\text{tr}(A + B) = \text{tr}A + \text{tr}B$ and $\text{tr}(cA) = c\,\text{tr}A$ for any scalar c.

c. Prove that $\text{tr}(AB) = \text{tr}(BA)$. (*Hint:* $\sum_{k=1}^{n}\sum_{\ell=1}^{n} c_{k\ell} = \sum_{\ell=1}^{n}\sum_{k=1}^{n} c_{k\ell}$.)

d. If $A, B \in \mathcal{M}_{n\times n}$, define $\langle A, B\rangle = \text{tr}(A^{\mathsf{T}}B)$. Check that this is an inner product on $\mathcal{M}_{n\times n}$.

e. Check that if A is symmetric and B is skew-symmetric, then $\langle A, B\rangle = 0$. (*Hint:* Use the properties to show that $\langle A, B\rangle = -\langle B, A\rangle$.)

f. Deduce that the subspaces of symmetric and skew-symmetric matrices (see Exercise 8) are orthogonal complements in $\mathcal{M}_{n\times n}$.

10. (See Exercise 9 for the relevant definitions.) Define $V = \{A \in \mathcal{M}_{n\times n} : \text{tr}A = 0\}$.

a. Show that V is a subspace of $\mathcal{M}_{n\times n}$.

*b. Find a basis for $V^{\perp}$ (using the inner product defined in Exercise 9).

11. Here is a sketch of the algebra result mentioned in the text. Let p be a polynomial of degree k, that is, $p(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$, where $a_0, \dots, a_k \in \mathbb{R}$ and $a_k \neq 0$.

a. Prove the *root-factor theorem*: c is a root of p, i.e., $p(c) = 0$, if and only if $p(t) = (t - c)q(t)$ for some polynomial q of degree $k - 1$. (*Hint:* When you divide $p(t)$ by $t - c$, the remainder should be $p(c)$. Why?)

b. Show that p has at most k roots.

12. a. Suppose b, c, and d are distinct real numbers. Show that the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{bmatrix}$$

is nonsingular.

b. Suppose a, b, c, and d are distinct real numbers. Show that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

is nonsingular. (*Hint:* Subtract a times the third row from the fourth, a times the second row from the third, and a times the first row from the second.)

c. Suppose $t_1, \dots, t_{k+1}$ are distinct. Prove that the matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_{k+1} \\ t_1^2 & t_2^2 & \dots & t_{k+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^k & t_2^k & \dots & t_{k+1}^k \end{bmatrix}$$

is nonsingular. (*Hint:* Iterate the trick from part b. If you know mathematical induction, this would be a good place to try it.)

13. a. Prove that $\dim \mathcal{M}_{n\times n} = n^2$.

b. Let $A \in \mathcal{M}_{n\times n}$. Show that there are scalars $c_0, c_1, \dots, c_{n^2}$, not all 0, so that $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_{n^2} A^{n^2} = O$. (That is, there is a nonzero polynomial p of degree at most n^2 so that $p(A) = O$.)

14. Let $g(t) = 1$. Using the inner product defined in Example 10(c), find the orthogonal complement of Span (g) in

*a. $\mathcal{P}_1 \subset \mathcal{C}^0([-1, 1])$

*b. $\mathcal{P}_1 \subset \mathcal{C}^0([0, 1])$

c. $\mathcal{P}_2 \subset \mathcal{C}^0([-1, 1])$

d. $\mathcal{P}_2 \subset \mathcal{C}^0([0, 1])$

15. Let $g_1(t) = 1$ and $g_2(t) = t$. Using the inner product defined in Example 10(c), find the orthogonal complement of Span (g_1, g_2) in

a. $\mathcal{P}_2 \subset \mathcal{C}^0([-1, 1])$

*b. $\mathcal{P}_2 \subset \mathcal{C}^0([0, 1])$

c. $\mathcal{P}_3 \subset \mathcal{C}^0([-1, 1])$

d. $\mathcal{P}_3 \subset \mathcal{C}^0([0, 1])$

***16.** Let $g_1(t) = t - 1$ and $g_2(t) = t^2 + t$. Using the inner product on $\mathcal{P}_2 \subset \mathcal{C}^0([0, 1])$ defined in Example 10(c), find the orthogonal complement of Span (g_1, g_2).

17. Let $g_1(t) = t - 1$ and $g_2(t) = t^2 + t$. Using the inner product on $\mathcal{P}_2$ defined in Example 10(b) with $t_1 = -1$, $t_2 = 0$, and $t_3 = 1$, find a basis for the orthogonal complement of Span (g_1, g_2).

***18.** Show that for any positive integer n, the functions $1, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt$ are orthogonal in $\mathcal{C}^\infty([-\pi, \pi]) \subset \mathcal{C}^0([-\pi, \pi])$ (using the inner product defined in Example 10(c)).

19. Using the inner product defined in Example 10(c), let $V = \mathcal{C}^0([a,b])$, and let $W = \{f \in V : \int_a^b f(t)\,dt = 0\}$.

a. Prove that W is a subspace of V.

b. Prove that $W^\perp$ is the subspace of constant functions.

c. Prove or disprove: $W + W^\perp = V$.

20. Prove that the following are subspaces of $\mathbb{R}^\omega$.

*a. $\{\mathbf{x} : \text{there is a constant } C \text{ such that } |x_k| \le C \text{ for all } k\}$

b. $\{\mathbf{x} : \lim_{k\to\infty} x_k \text{ exists}\}$

c. $\{\mathbf{x} : \lim_{k\to\infty} x_k = 0\}$

d. $\{\mathbf{x} : \sum_{k=1}^{\infty} x_k \text{ exists}\}$

e. $\{\mathbf{x} : \sum_{k=1}^{\infty} |x_k| \text{ exists}\}$ (*Hint:* Remember that $|a+b| \le |a| + |b|$ for all $a, b \in \mathbb{R}$.)

21. The subspace $\ell^2 \subset \mathbb{R}^\omega$ defined by $\ell^2 = \{\mathbf{x} \in \mathbb{R}^\omega : \sum_{k=1}^{\infty} x_k^2 \text{ exists}\}$ is an inner product space with inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{\infty} x_k y_k$. (That this sum makes sense follows by "taking the limit" of the Cauchy-Schwarz Inequality.) Let

$$V = \{\mathbf{x} \in \ell^2 : \text{there is an integer } n \text{ such that } x_k = 0 \text{ for all } k > n\}.$$

Show that V is a (proper) subspace of ℓ^2 and that $V^\perp = \{\mathbf{0}\}$. It follows that $(V^\perp)^\perp = \ell^2$, so Proposition 3.6 need not hold in infinite-dimensional spaces.

HISTORICAL NOTES

This chapter begins with the vectors you studied in the first chapters and puts them in an algebraic setting, that of a vector space. The chapter ends with another extension of these ideas, the formal definition of an abstract vector space. This definition has its origins in the 1888 publication *Geometrical Calculus* by the Italian mathematician Giuseppe Peano (1858–1932). Peano's forte was formulating precise definitions and axioms in various areas of mathematics and producing rigorous proofs of his mathematical assertions. Because of his extremely careful approach to mathematics, he often found errors in other mathematicians' work and occasionally found himself in heated arguments with his contemporaries on the importance of such mathematical rigor. He is particularly known for his foundational work in mathematics and his development of much of the concomitant notation. His definition of an abstract vector space in 1888 shows his penchant for precision and is essentially what we use today.

Although formalism is extremely important in mathematics and Peano's work should be considered ahead of its time, the historical origins of the vector space lie with those who first discovered and exploited its essential properties. The ideas of linear combinations arose early in the study of differential equations. The history of the latter is itself a fascinating topic, with a great deal of activity beginning in the seventeenth century and continuing to the present day. The idea that a linear combination of solutions of a linear ordinary differential equation is itself a solution can be found in a 1739 letter from Leonhard Euler (1707–1783) to Johann Bernoulli (1667–1748). Of course, this means that the collection of *all* solutions forms a vector space, but Euler didn't use that term. The notions of linear independence and basis also emerge in that letter, as Euler discusses writing the general solution of the differential equation as a linear combination of certain base solutions. These ideas continued to show up in works of other great mathematicians who studied differential equations, notably Jean le Rond d'Alembert (1717–1783) and Joseph-Louis Lagrange (1736–1813). Not long thereafter, the ideas of vector space and dimension found their way into the study of geometry with the work of Hermann Gunther Grassmann (1809–1877). In 1844 he published a seminal work describing his "calculus of extension," now called exterior algebra. Because so many people found his work unreadable, he ultimately revised it and published *Die Ausdehnungslehre* in 1862. The objects he introduced are linear combinations of symbols representing points, lines, and planes in various dimensions. In fact, it was Grassmann's work that inspired Peano to make the modern definitions of basis and dimensions.

The history of dimension is itself a fascinating subject. For vector spaces, we have seen that the definition is fairly intuitive. On the other hand, there are some very interesting paradoxes that arise when one examines the notion of dimension more carefully. In 1877 Georg Cantor (1845–1918) made an amazing and troubling discovery: He proved that that there is a one-to-one correspondence between points of $\mathbb{R}$ and points of $\mathbb{R}^2$. Although intuitively the (two-dimensional) plane is bigger than the (one-dimensional) line, Cantor's argument says that they each have the same "number" of points.

Of course, the one-to-one correspondence described by Cantor must not be a very nice function, so one might consider Cantor's result not as a problem with the concept of dimension but as evidence that perhaps functions can be badly behaved. However, in 1890 Peano further confounded the situation by producing a *continuous* function from the line into the plane that touches every point of a square. The actual definition of dimension that finally came about to resolve these issues is too involved for us to discuss here, as it entails elements of the branch of mathematics known as topology and objects called manifolds. The key players in this work were Georg Friedrich Bernhard Riemann (1826–1866), J. Henri Poincaré (1854–1912), and L. E. J. Brouwer (1881–1966).

Even today, new discoveries and definitions shatter people's expectations of dimension. Cantor and Peano did not realize that they were laying the groundwork for what is now the study of *fractals* or fractal geometry. Benoît Mandelbrot (1924–) coined the term *fractal* (from the Latin *fractus*, describing the appearance of a broken stone as irregular and fragmented) and launched a fascinating area of study. Fractal curves are strange beasts in that they always appear the same no matter how closely you look at them. When the notions of dimension were extended to capture this behavior, mathematicians had discovered geometric figures with fractional dimension. Indeed, the coastline of Britain could actually be considered to have fractal dimension approximately 1.2. The study of fractals is a very active field of research today.

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CHAPTER 4

PROJECTIONS AND LINEAR TRANSFORMATIONS

An inconsistent linear system $A\mathbf{x} = \mathbf{b}$ has, of course, no solution. But we might try to do the best we can by solving instead the system $A\mathbf{x} = \mathbf{p}$, where $\mathbf{p}$ is the vector closest to $\mathbf{b}$ lying in the column space of A. This naturally involves the notion of projection of vectors in $\mathbb{R}^n$ onto subspaces, which is our fundamental example of a linear transformation. In this chapter, we continue the discussion, initiated in Chapter 2, of linear transformations—the functions underlying matrices. We will see that a given linear transformation can be represented by quite different matrices, depending on the underlying basis (or coordinate system) for our vector space. A propitious choice of basis can lead to a more "convenient" matrix representation and a better understanding of the linear transformation itself.

1 Inconsistent Systems and Projection

Suppose we're given the system $A\mathbf{x} = \mathbf{b}$ to solve, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

As the reader can check, $\mathbf{b} \notin \mathbf{C}(A)$, and so this system is inconsistent. It seems reasonable instead to solve $A\mathbf{x} = \mathbf{p}$, where $\mathbf{p}$ is the vector in $\mathbf{C}(A)$ that is *closest* to $\mathbf{b}$. The vector $\mathbf{p}$ is characterized by the following lemma.

Lemma 1.1. *Suppose $V \subset \mathbb{R}^m$ is a subspace, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{p} \in V$ has the property that $\mathbf{b} - \mathbf{p} \in V^\perp$. Then*

$$\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{q}\| \quad \text{for all other vectors } \mathbf{q} \in V.$$

That is, if $\mathbf{p} \in V$ and $\mathbf{p}$ differs from $\mathbf{b}$ by an element of $V^\perp$, then $\mathbf{p}$ is closer to $\mathbf{b}$ than every other point in V.

Proof. Since $\mathbf{b}-\mathbf{p} \in V^{\perp}$ and $\mathbf{q}-\mathbf{p} \in V$, the vectors $\mathbf{q}-\mathbf{p}$ and $\mathbf{b}-\mathbf{p}$ are orthogonal and form a right triangle with hypotenuse $\mathbf{b}-\mathbf{q}$, as shown in Figure 1.1. Thus, by the Pythagorean Theorem, $\|\mathbf{b}-\mathbf{q}\| > \|\mathbf{b}-\mathbf{p}\|$, and so $\mathbf{p}$ is closer to $\mathbf{b}$ than *every* other point in V. □

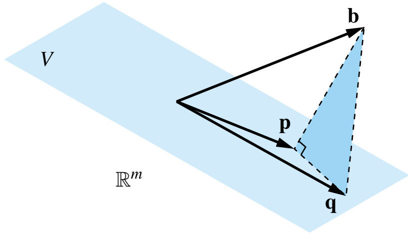

FIGURE 1.1

Recall that Theorem 4.9 of Chapter 3 tells us that for any subspace $V \subset \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^m$, there is a unique way to write

$$\boxed{\mathbf{b} = \mathbf{p} + (\mathbf{b}-\mathbf{p}), \qquad \text{where} \quad \mathbf{p} \in V \text{ and } \mathbf{b}-\mathbf{p} \in V^{\perp}.}$$

This leads to the following definition.

Definition. Let $V \subset \mathbb{R}^m$ be a subspace, and let $\mathbf{b} \in \mathbb{R}^m$. We define the *projection of* $\mathbf{b}$ onto V to be the unique vector $\mathbf{p} \in V$ with the property that $\mathbf{b}-\mathbf{p} \in V^{\perp}$. We write $\mathbf{p} = \text{proj}_V \mathbf{b}$.

Remark. Since $(V^{\perp})^{\perp} = V$, it follows that $\text{proj}_{V^{\perp}}\mathbf{b} = \mathbf{b}-\mathbf{p} = \mathbf{b} - \text{proj}_V \mathbf{b}$. (Note that $\mathbf{b}-\mathbf{p} \in V^{\perp}$ and $\mathbf{b}-(\mathbf{b}-\mathbf{p}) = \mathbf{p} \in (V^{\perp})^{\perp}$.) In particular,

$$\boxed{\mathbf{b} = \text{proj}_V \mathbf{b} + \text{proj}_{V^{\perp}}\mathbf{b}.}$$

To calculate $\mathbf{p}$ explicitly, we proceed as follows: Suppose V is n-dimensional and choose vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ that form a basis for V. Consider the $m \times n$ matrix A whose columns are $\mathbf{v}_1, \dots, \mathbf{v}_n$. We want to solve

$$A\overline{\mathbf{x}} = \mathbf{p} \quad \text{where} \quad \mathbf{b}-\mathbf{p} \in \mathbf{C}(A)^{\perp}.$$

By Theorem 2.5 of Chapter 3, $\mathbf{C}(A)^{\perp} = \mathbf{N}(A^{\mathsf{T}})$, so now we can rewrite our problem as a pair of equations:

$$\begin{aligned} A\overline{\mathbf{x}} &= \mathbf{p} \\ A^{\mathsf{T}}(\mathbf{b}-\mathbf{p}) &= \mathbf{0}. \end{aligned}$$

Substituting the first in the second yields

$$A^{\mathsf{T}}(\mathbf{b}-A\overline{\mathbf{x}}) = \mathbf{0}.$$

That is, $\overline{\mathbf{x}}$ is a solution of the *normal equations*[1]

$$\boxed{A^{\mathsf{T}}A\overline{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}.}$$

We already know this equation has (at least) a solution, because $\mathbf{p}$ exists. Since, by construction, the rank of A is n, we claim that $A^{\mathsf{T}}A$ is nonsingular: Suppose $\mathbf{x} \in \mathbf{N}(A^{\mathsf{T}}A)$.

[1] The nomenclature is yet another example of the lively imagination of mathematicians.

This means

$$(A^{\mathsf{T}}A)\mathbf{x} = A^{\mathsf{T}}(A\mathbf{x}) = \mathbf{0},$$

so, dotting with $\mathbf{x}$, we get

$$A^{\mathsf{T}}(A\mathbf{x}) \cdot \mathbf{x} = A\mathbf{x} \cdot A\mathbf{x} = \|A\mathbf{x}\|^2 = 0;$$

therefore $A\mathbf{x} = \mathbf{0}$, and since A has rank n, we conclude that $\mathbf{x} = \mathbf{0}$. As a result, we have the following definition.

Definition. Given an $m \times n$ matrix A of rank n, the *least squares solution* of the equation $A\mathbf{x} = \mathbf{b}$ is the unique solution[2] of the normal equations

$$(A^{\mathsf{T}}A)\overline{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}.$$

Remark. When $\operatorname{rank}(A) < n$, the normal equations are still consistent (see Exercise 3.4.24) but have infinitely many solutions. Every one of those solutions $\mathbf{x}$ gives rise to the same projection $A\mathbf{x} = \mathbf{p}$, because $\mathbf{p}$ is the unique vector in $\mathbf{C}(A)$ with the property that $\mathbf{b} - \mathbf{p} \in \mathbf{N}(A^{\mathsf{T}})$. But we define *the* least squares solution to be the unique vector $\overline{\mathbf{x}} \in \mathbf{R}(A)$ that satisfies the normal equations; of all the solutions, this is the one of least length. (See Proposition 4.10 of Chapter 3.) This leads to the *pseudoinverse* that is important in numerical analysis. See Strang's books for more details.

Summarizing, we have proved the following proposition.

Proposition 1.2. *Suppose $V \subset \mathbb{R}^m$ is an n-dimensional subspace with basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let A be the $m \times n$ matrix whose columns are these basis vectors. Given a vector $\mathbf{b} \in \mathbb{R}^m$, the projection of $\mathbf{b}$ onto V is obtained by multiplying the unique solution of the normal equations $(A^{\mathsf{T}}A)\overline{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}$ by the matrix A. That is,*

$$\mathbf{p} = \operatorname{proj}_V \mathbf{b} = A\overline{\mathbf{x}} = \big(A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\big)\mathbf{b}.$$

EXAMPLE 1

To solve the problem we posed at the beginning of the section, we wish to find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

We need only solve the normal equations $A^{\mathsf{T}}A\overline{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}$. Now,

$$A^{\mathsf{T}}A = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^{\mathsf{T}}\mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

[2] $\overline{\mathbf{x}}$ is called the *least squares solution* because it minimizes the sum of squares $\|\mathbf{b} - \mathbf{q}\|^2 = (b_1 - q_1)^2 + \cdots + (b_m - q_m)^2$, among all vectors $\mathbf{q} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

and so, using the formula for the inverse of a 2×2 matrix in Example 4 on p. 105, we find that

$$\overline{\mathbf{x}} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\mathbf{b} = \frac{1}{20}\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

is the least squares solution. Moreover, while we're at it, we can find the projection of $\mathbf{b}$ onto $\mathbf{C}(A)$ by calculating

$$A\overline{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}\left(\frac{1}{10}\begin{bmatrix} 3 \\ 11 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} 17 \\ 14 \\ 11 \\ -8 \end{bmatrix}.$$

▲

We note next that Proposition 1.2 gives us an explicit formula for projection onto a subspace $V \subset \mathbb{R}^m$, generalizing the case of $\dim V = 1$ given in Chapter 1. Since we have

$$\mathbf{p} = \text{proj}_V \mathbf{b} = \big(A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\big)\mathbf{b}$$

for every vector $\mathbf{b} \in \mathbb{R}^m$, it follows that the matrix

$$(*) \qquad P_V = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}$$

is the appropriate "projection matrix"; that is,

$$\text{proj}_V = \mu_{P_V} = \mu_{A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}}.$$

Remark. It can be rather messy to compute P_V, since along the way we must calculate $(A^{\mathsf{T}}A)^{-1}$. Later in this chapter, we will see that a clever choice of basis will make this calculation much easier.

Remark. Since we have written $\text{proj}_V = \mu_{P_V}$ for the projection matrix P_V, it follows that proj_V is a linear transformation. (We already checked this for one-dimensional subspaces V in Chapter 2.) In Exercise 13 we ask the reader to show this directly, using the definition on p. 192.

EXAMPLE 2

If $\mathbf{b} \in \mathbf{C}(A)$ to begin with, then $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, and

$$P_V\mathbf{b} = \big(A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\big)\mathbf{b} = A(A^{\mathsf{T}}A)^{-1}(A^{\mathsf{T}}A)\mathbf{x} = A\mathbf{x} = \mathbf{b},$$

as it should be. And if $\mathbf{b} \in \mathbf{C}(A)^{\perp}$, then $\mathbf{b} \in \mathbf{N}(A^{\mathsf{T}})$, so

$$P_V\mathbf{b} = \big(A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}\big)\mathbf{b} = A(A^{\mathsf{T}}A)^{-1}(A^{\mathsf{T}}\mathbf{b}) = \mathbf{0},$$

as it should be. ▲

EXAMPLE 3

Note that when $\dim V = 1$, we recover our formula for projection onto a line from Section 2 of Chapter 1. If $\mathbf{a} \in V \subset \mathbb{R}^m$ is a nonzero vector, we consider it as an $m \times 1$ matrix, and the projection formula becomes

$$P_V = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^{\mathsf{T}};$$

that is,

$$P_V\mathbf{b} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{a}\mathbf{a}^\mathsf{T})\mathbf{b} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}(\mathbf{a}^\mathsf{T}\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a},$$

as before. For example, if

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{then} \quad P_V = \frac{1}{5}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(This might be a good time to review Example 3 in Section 5 of Chapter 2. For $\mathbf{a} \in \mathbb{R}^m$, remember that $\mathbf{a}\mathbf{a}^\mathsf{T}$ is an $m \times m$ matrix, whereas $\mathbf{a}^\mathsf{T}\mathbf{a}$ is a 1×1 matrix, i.e., a scalar.) ▲

EXAMPLE 4

Let $V \subset \mathbb{R}^3$ be the plane defined by the equation $x_1 - 2x_2 + x_3 = 0$. Then

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for V, and we take

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, since

$$A^\mathsf{T}A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}, \quad \text{we have} \quad (A^\mathsf{T}A)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix},$$

and so

$$P_V = A(A^\mathsf{T}A)^{-1}A^\mathsf{T} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

Here is an alternative solution. We have already seen that

$$\operatorname{proj}_V \mathbf{b} = \mathbf{b} - \operatorname{proj}_{(V^\perp)}\mathbf{b}.$$

In matrix notation, this can be rewritten as

$$P_V = I_3 - P_{V^\perp}.$$

Since $V^\perp$ is spanned by a normal vector, $\mathbf{a}$, to the plane, as in Example 3, we have

$$P_{V^\perp} = \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^\mathsf{T}, \quad \text{and so} \quad P_V = I_3 - \frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^\mathsf{T}.$$

In our case, we have $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and so

$$I_3 - \frac{1}{\|\mathbf{a}\|^2}(\mathbf{a}\mathbf{a}^{\mathsf{T}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix},$$

just as before. This is a useful trick to keep in mind. It is sometimes easier to calculate $P_{V^\perp}$ than P_V; generally, it is easier to project onto the subspace that has the smaller dimension. ▲

In the next section, we'll see the geometry underlying the formula for the projection matrix.

1.1 Data Fitting

Perhaps the most natural setting in which inconsistent systems of linear equations arise is that of fitting data to a linear model when they won't quite fit. The least squares solution of such linear problems is called the *least squares line* fitting the points (or the *line of regression* in statistics). Even nonlinear problems can sometimes be rephrased linearly. For example, fitting a parabola $y = ax^2 + bx + c$ to data points in the xy-plane, as we saw in Section 6 of Chapter 1, is still a matter of solving a system of linear equations. Even more surprisingly, in our laboratory work many of us have tried to find the right constants a and k so that the data points $(x_1, y_1), \dots, (x_m, y_m)$ lie on the curve $y = ax^k$. Taking logarithms, we see that this is equivalent to fitting the points $(u_i, v_i) = (\ln x_i, \ln y_i)$, $i = 1, \dots, m$, to a *line* $v = ku + \ln a$.[3]

EXAMPLE 5

Find the least squares line $y = ax + b$ for the data points $(-1, 0)$, $(1, 1)$, and $(2, 3)$. (See Figure 1.2.) We get the system of equations

$$\begin{aligned} -1a + b &= 0 \\ 1a + b &= 1 \\ 2a + b &= 3, \end{aligned}$$

which in matrix form becomes

$$A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

[3]This is why "log-log paper" was so useful before the advent of calculators and computers.

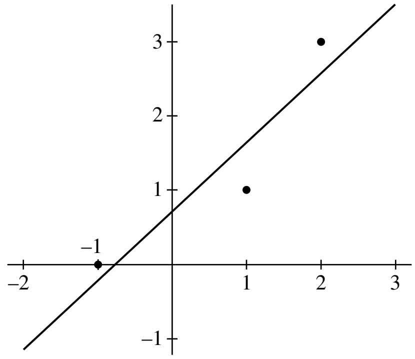

FIGURE 1.2

The least squares solution is

$$\begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 13 \\ 10 \end{bmatrix}.$$

That is, the least squares line is

$$y = \frac{13}{14}x + \frac{5}{7}.$$

▲

When we find the least squares line $y = \overline{a}x + \overline{b}$ fitting the data points $(x_1, y_1), \dots, (x_m, y_m)$, we are finding the least squares solution of the (inconsistent) system $A\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{y}$, where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

Let's denote by $\overline{\mathbf{y}} = A\begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix}$ the projection of $\mathbf{y}$ onto $\mathbf{C}(A)$. The least squares solution $\begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix}$ has the property that $\|\mathbf{y} - \overline{\mathbf{y}}\|$ is as small as possible, whence the name *least squares*. If we define the *error* vector $\boldsymbol{\epsilon} = \mathbf{y} - \overline{\mathbf{y}}$, then we have

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{bmatrix} = \begin{bmatrix} y_1 - \overline{y}_1 \\ y_2 - \overline{y}_2 \\ \vdots \\ y_m - \overline{y}_m \end{bmatrix} = \begin{bmatrix} y_1 - (\overline{a}x_1 + \overline{b}) \\ y_2 - (\overline{a}x_2 + \overline{b}) \\ \vdots \\ y_m - (\overline{a}x_m + \overline{b}) \end{bmatrix}.$$

The least squares process chooses $\overline{a}$ and $\overline{b}$ so that $\|\boldsymbol{\epsilon}\|^2 = \epsilon_1^2 + \cdots + \epsilon_m^2$ is as small as possible. But something interesting happens. Recall that

$$\boldsymbol{\epsilon} = \mathbf{y} - \overline{\mathbf{y}} \in \mathbf{C}(A)^{\perp}.$$

Thus, $\boldsymbol{\epsilon}$ is orthogonal to each of the column vectors of A, and so, in particular,

$$\begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_m \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \epsilon_1 + \cdots + \epsilon_m = 0.$$

That is, in the process of minimizing the sum of the squares of the errors ϵ_i, we have in fact made their (algebraic) sum equal to 0.

Exercises 4.1

1. By first finding the projection onto $V^\perp$, find the projection of the given vector $\mathbf{b} \in \mathbb{R}^m$ onto the given hyperplane $V \subset \mathbb{R}^m$.

a. $V = \{x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$, $\mathbf{b} = (2, 1, 1)$

*b. $V = \{x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^4$, $\mathbf{b} = (0, 1, 2, 3)$

c. $V = \{x_1 - x_2 + x_3 + 2x_4 = 0\} \subset \mathbb{R}^4$, $\mathbf{b} = (1, 1, 1, 1)$

2. Use the formula $P_V = A(A^\top A)^{-1}A^\top$ for the projection matrix to check that $P_V = P_V^\top$ and $P_V^2 = P_V$. Show that $I - P_V$ has the same properties, and explain why.

*__3.__ Let $V = \text{Span}\big((1, 1, -1), (-2, 0, 1)\big) \subset \mathbb{R}^3$. Construct the matrix P_V representing proj_V

a. by finding $P_{V^\perp}$;

b. by using the formula $(*)$ on p. 194.

4. Let $V = \text{Span}\big((1, 0, 1), (0, 1, -2)\big) \subset \mathbb{R}^3$. Construct the matrix P_V representing proj_V

a. by finding $P_{V^\perp}$;

b. by using the formula $(*)$ on p. 194.

5. Let $V = \text{Span}\big((1, 0, 1, 0), (0, 1, 0, 1), (1, 1, -1, -1)\big) \subset \mathbb{R}^4$. Construct the matrix P_V representing proj_V

a. by finding $P_{V^\perp}$;

b. by using the formula $(*)$ on p. 194.

*__6.__ Find the least squares solution of

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + x_2 &= -2 \\ x_1 - x_2 &= 1\,. \end{aligned}$$

Use your answer to find the point on the plane spanned by $(1, 2, 1)$ and $(1, 1, -1)$ that is closest to $(4, -2, 1)$.

7. Find the least squares solution of

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - 3x_2 &= 4 \\ 2x_1 + x_2 &= 3\,. \end{aligned}$$

Use your answer to find the point on the plane spanned by $(1, 1, 2)$ and $(1, -3, 1)$ that is closest to $(1, 4, 3)$.

8. Find the least squares solution of

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 &= 4 \\ x_1 + 2x_2 &= 3\,. \end{aligned}$$

Use your answer to find the point on the plane spanned by (1, 1, 1) and (−1, 0, 2) that is closest to (1, 4, 3).

9. Consider the four data points (−1, 0), (0, 1), (1, 3), (2, 5).

*a. Find the "least squares horizontal line" $y = a$ fitting the data points. Check that the sum of the errors is 0.

b. Find the "least squares line" $y = ax + b$ fitting the data points. Check that the sum of the errors is 0.

*c. (Calculator recommended) Find the "least squares parabola" $y = ax^2 + bx + c$ fitting the data points. What is true of the sum of the errors in this case?

10. Consider the four data points (1, 1), (2, 2), (3, 1), (4, 3).

a. Find the "least squares horizontal line" $y = a$ fitting the data points. Check that the sum of the errors is 0.

b. Find the "least squares line" $y = ax + b$ fitting the data points. Check that the sum of the errors is 0.

c. (Calculator recommended) Find the "least squares parabola" $y = ax^2 + bx + c$ fitting the data points. What is true of the sum of the errors in this case?

***11.** (Calculator required) Find the least squares fit of the form $y = ax^k$ to the data points (1, 2), (2, 3), (3, 5), and (5, 8).

12. Given data points $(x_1, y_1), \dots, (x_m, y_m)$, let $(x_1, \overline{y}_1), \dots, (x_m, \overline{y}_m)$ be the corresponding points on the least squares line.

a. Show that $\overline{y}_1 + \cdots + \overline{y}_m = y_1 + \cdots + y_m$.

b. Conclude that the least squares line passes through the centroid $\left(\frac{1}{m}\sum_{i=1}^{m} x_i,\ \frac{1}{m}\sum_{i=1}^{m} y_i\right)$ of the original data points.

c. Show that $\sum_{i=1}^{m} x_i\overline{y}_i = \sum_{i=1}^{m} x_i y_i$. (*Hint:* This is one of the few times that the preceding parts of the exercise are not relevant.)

***13.** Use the *definition* of projection on p. 192 to show that for any subspace $V \subset \mathbb{R}^m$, $\text{proj}_V : \mathbb{R}^m \to \mathbb{R}^m$ is a linear transformation. That is, show that

a. $\text{proj}_V(\mathbf{x} + \mathbf{y}) = \text{proj}_V\mathbf{x} + \text{proj}_V\mathbf{y}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$;

b. $\text{proj}_V(c\mathbf{x}) = c\ \text{proj}_V\mathbf{x}$ for all vectors $\mathbf{x}$ and scalars c.

14. Prove from the *definition* of projection on p. 192 that if $\text{proj}_V = \mu_A$, then $A = A^2$ and $A = A^{\mathsf{T}}$. (*Hint:* For the latter, show that $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. It may be helpful to write $\mathbf{x}$ and $\mathbf{y}$ as the sum of vectors in V and $V^{\perp}$. Then use Exercise 2.5.24.)

15. Prove that if $A^2 = A$ and $A = A^{\mathsf{T}}$, then A is a projection matrix. (*Hints:* First decide onto which subspace it should be projecting. Then show that for all $\mathbf{x}$, the vector $A\mathbf{x}$ lies in that subspace and $\mathbf{x} - A\mathbf{x}$ is orthogonal to that subspace.)

16. Let V and W be subspaces of $\mathbb{R}^n$ and let $\mathbf{b} \in \mathbb{R}^n$. The *affine subspace* passing through $\mathbf{b}$ and parallel to V is defined to be

$$\mathbf{b} + V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{b} + \mathbf{v} \text{ for some } \mathbf{v} \in V\}.$$

a. Suppose $\mathbf{x} \in \mathbf{b} + V$ and $\mathbf{y} \in \mathbf{c} + W$ have the property that $\mathbf{x} - \mathbf{y}$ is orthogonal to both V and W. Show that $\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}' - \mathbf{y}'\|$ for any $\mathbf{x}' \in \mathbf{b} + V$ and $\mathbf{y}' \in \mathbf{c} + W$. (Thus, $\mathbf{x}$ and $\mathbf{y}$ are the points in $\mathbf{b} + V$ and $\mathbf{c} + W$, respectively, that are closest.)

b. Show that the distance between the affine subspaces $\mathbf{b} + V$ and $\mathbf{c} + W$ (see Figure 1.3), i.e., the least possible distance between a point in one and a point in the other, is

$$\|\text{proj}_{(V+W)^\perp}(\mathbf{b} - \mathbf{c})\|.$$

c. Deduce that this distance is 0 when $V + W = \mathbb{R}^n$. Give the obvious geometric explanation.

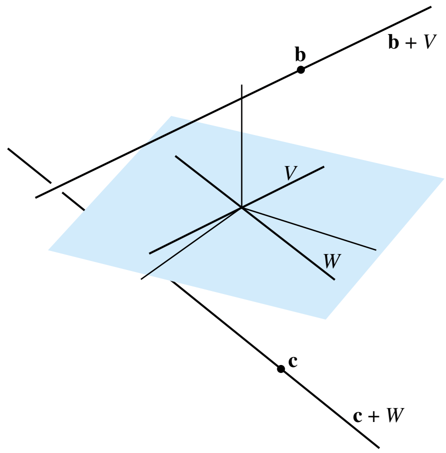

FIGURE 1.3

17. Using the formula derived in Exercise 16, find the distance

*a. between the skew lines ℓ: $(2, 1, 1) + t(0, 1, -1)$ and m: $(1, 1, 0) + s(1, 1, 1)$ in $\mathbb{R}^3$

b. between the skew lines ℓ: $(1, 1, 1, 0) + t(1, 0, 1, 1)$ and m: $(0, 0, 0, 1) + s(1, 1, 0, 2)$ in $\mathbb{R}^4$

c. between the line ℓ: $(1, 0, 0, 1) + t(1, 0, 1, 0)$ and the 2-dimensional plane $\mathcal{P} = \text{Span}\big((1, 1, 1, 1), (0, 1, 1, 2)\big)$ in $\mathbb{R}^4$

2 Orthogonal Bases

We saw in the last section how to find the projection of a vector onto a subspace $V \subset \mathbb{R}^m$ using the so-called normal equations. But the inner workings of the formula $(*)$ on p. 194 are still rather mysterious. Because we have known since Chapter 1 how to project a vector $\mathbf{x}$ onto a line, it might seem more natural to start with a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for V and sum up the projections of $\mathbf{x}$ onto the $\mathbf{v}_j$'s. However, as we see in the diagram on the left in Figure 2.1, when we start with $\mathbf{x} \in V$ and add the projections of $\mathbf{x}$ onto the vectors of an arbitrary basis for V, the resulting vector needn't have much to do with $\mathbf{x}$. Nevertheless, the diagram on the right suggests that when we start with a basis consisting of mutually orthogonal vectors, the process may work. We begin by proving this as a lemma.

Definition. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$. We say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an *orthogonal set* of vectors provided $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. We say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an *orthogonal basis* for a subspace V if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is both a basis for V and an orthogonal set. Moreover, we say $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an *orthonormal basis* for V if it is an orthogonal basis consisting of unit vectors.

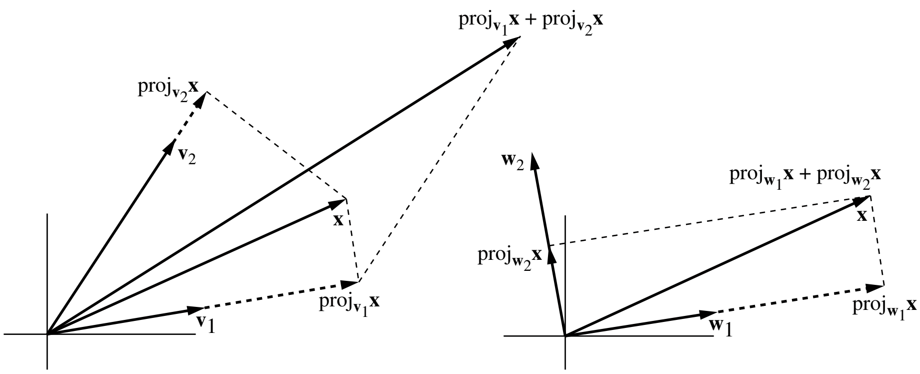

FIGURE 2.1

The first reason that orthogonal sets of vectors are so important is the following:

Proposition 2.1. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$. *If* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is an orthogonal set of* nonzero *vectors, then* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

Proof. This was the content of Exercise 3.3.10, but the result is so important that we give the proof here. Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

We must show that $c_1 = c_2 = \cdots = c_k = 0$. For any $i = 1, \dots, k$, we take the dot product of this equation with $\mathbf{v}_i$, obtaining

$$c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \cdots + c_i(\mathbf{v}_i \cdot \mathbf{v}_i) + \cdots + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) = 0,$$

from which we see that $c_i\|\mathbf{v}_i\|^2 = 0$. Since $\mathbf{v}_i \neq \mathbf{0}$, it follows that $c_i = 0$. Because this holds for any $i = 1, \dots, k$, we have $c_1 = c_2 = \cdots = c_k = 0$, as required. □

The same sort of calculation shows us how to write a vector as a linear combination of orthogonal basis vectors, as we saw in Exercise 3.3.11.

Lemma 2.2. *Suppose* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is a basis for* V. *Then the equation*

$$\mathbf{x} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i}\mathbf{x} = \sum_{i=1}^{k} \frac{\mathbf{x} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}\mathbf{v}_i$$

holds for all $\mathbf{x} \in V$ *if and only if* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is an orthogonal basis for* V.

Proof. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for V. Then there are scalars $c_1, \dots, c_k$ so that

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i + \cdots + c_k\mathbf{v}_k.$$

Taking advantage of the orthogonality of the $\mathbf{v}_j$'s, we take the dot product of this equation with $\mathbf{v}_i$:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{v}_i &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \cdots + c_i(\mathbf{v}_i \cdot \mathbf{v}_i) + \cdots + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) \\ &= c_i\|\mathbf{v}_i\|^2,\end{aligned}$$

and so

$$c_i = \frac{\mathbf{x} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}.$$

(Note that $\mathbf{v}_i \neq \mathbf{0}$ because $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ forms a basis for V.)

Conversely, suppose that every vector $\mathbf{x} \in V$ is the sum of its projections on $\mathbf{v}_1, \dots, \mathbf{v}_k$. Let's just examine what this means when $\mathbf{x} = \mathbf{v}_1$: We are given that

$$\mathbf{v}_1 = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{v}_1 = \sum_{i=1}^{k} \frac{\mathbf{v}_1 \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i.$$

Recall from Proposition 3.1 of Chapter 3 that every vector has a *unique* expansion as a linear combination of basis vectors, so comparing coefficients of $\mathbf{v}_2, \dots, \mathbf{v}_k$ on either side of this equation, we conclude that

$$\mathbf{v}_1 \cdot \mathbf{v}_i = 0 \quad \text{for all } i = 2, \dots, k.$$

A similar argument shows that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, and the proof is complete. □

We recall that whenever $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for V, every vector $\mathbf{x} \in V$ can be written uniquely as a linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

where the coefficients $c_1, c_2, \dots, c_k$ are called the *coordinates* of $\mathbf{x}$ with respect to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. We emphasize that, as in Lemma 2.2, when $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ forms an *orthogonal basis* for V, the dot product gives the coordinates of $\mathbf{x}$. As we shall see in Section 4, when the basis is not orthogonal, it is more tedious to compute these coordinates.

Not only do orthogonal bases make it easy to calculate coordinates, they also make projections quite easy to compute, as we now see.

Proposition 2.3. *Let $V \subset \mathbb{R}^m$ be a k-dimensional subspace. The equation*

$$(\dagger) \qquad \text{proj}_V \mathbf{b} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b} = \sum_{i=1}^{k} \frac{\mathbf{b} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$$

holds for all $\mathbf{b} \in \mathbb{R}^m$ if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for V.

Proof. Assume $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for V and let $\mathbf{b} \in \mathbb{R}^m$. Write $\mathbf{b} = \mathbf{p} + (\mathbf{b} - \mathbf{p})$, where $\mathbf{p} = \text{proj}_V \mathbf{b}$ (and so $\mathbf{b} - \mathbf{p} \in V^\perp$). Then, since $\mathbf{p} \in V$, it follows from Lemma 2.2 that $\mathbf{p} = \sum_{i=1}^{k} \frac{\mathbf{p} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$. Moreover, for $i = 1, \dots, k$, we have $\mathbf{b} \cdot \mathbf{v}_i = \mathbf{p} \cdot \mathbf{v}_i$, since $\mathbf{b} - \mathbf{p} \in V^\perp$. Thus,

$$\text{proj}_V \mathbf{b} = \mathbf{p} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{p} = \sum_{i=1}^{k} \frac{\mathbf{p} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i = \sum_{i=1}^{k} \frac{\mathbf{b} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b}.$$

Conversely, suppose $\text{proj}_V \mathbf{b} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$. In particular, when $\mathbf{b} \in V$, we deduce that $\mathbf{b} = \text{proj}_V \mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$, so these vectors span V; since V is k-dimensional, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ gives a basis for V. By Lemma 2.2, it must be an orthogonal basis. □

We now have another way to calculate the projection of a vector on a subspace V, provided we can come up with an orthogonal basis for V.

EXAMPLE 1

We return to Example 4 of Section 1. The basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ we used there was certainly not an orthogonal basis, but it is not hard to come up with one that is. Instead, we take

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(It is immediate that $\mathbf{w}_1 \cdot \mathbf{w}_2 = 0$ and that $\mathbf{w}_1, \mathbf{w}_2$ lie in the plane $x_1 - 2x_2 + x_3 = 0$.) Now, we calculate

$$\begin{aligned}
\text{proj}_V \mathbf{b} = \text{proj}_{\mathbf{w}_1}\mathbf{b} + \text{proj}_{\mathbf{w}_2}\mathbf{b} &= \frac{\mathbf{b}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\mathbf{b}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 \\
&= \left(\frac{1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1\mathbf{w}_1^{\mathsf{T}} + \frac{1}{\|\mathbf{w}_2\|^2}\mathbf{w}_2\mathbf{w}_2^{\mathsf{T}}\right)\mathbf{b} \\
&= \left(\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)\mathbf{b} \\
&= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}\mathbf{b},
\end{aligned}$$

as we found earlier. ▲

Remark. This is exactly what we get from formula (∗) on p. 194 when $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal set. In particular,

$$\begin{aligned}
P_V &= A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}} \\
&= \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} \frac{1}{\|\mathbf{v}_1\|^2} & & & \\ & \frac{1}{\|\mathbf{v}_2\|^2} & & \\ & & \ddots & \\ & & & \frac{1}{\|\mathbf{v}_k\|^2} \end{bmatrix} \begin{bmatrix} \text{---} & \mathbf{v}_1^{\mathsf{T}} & \text{---} \\ \text{---} & \mathbf{v}_2^{\mathsf{T}} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_k^{\mathsf{T}} & \text{---} \end{bmatrix} \\
&= \sum_{i=1}^{k} \frac{1}{\|\mathbf{v}_i\|^2}\mathbf{v}_i\mathbf{v}_i^{\mathsf{T}}.
\end{aligned}$$

To see why the last equality holds, the reader can either apply both sides to a vector $\mathbf{x} \in \mathbb{R}^m$ or think through the usual procedure for multiplying matrices (preferably using one finger from each hand). See also Exercise 2.5.4.

Now it is time to develop an algorithm for transforming a given (ordered) basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for a subspace (or inner product space) into an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$, as shown in Figure 2.2. The idea is quite simple. We set

$$\mathbf{w}_1 = \mathbf{v}_1.$$

If $\mathbf{v}_2$ is orthogonal to $\mathbf{w}_1$, then we set $\mathbf{w}_2 = \mathbf{v}_2$. Of course, in general, it will not be, and we want $\mathbf{w}_2$ to be the part of $\mathbf{v}_2$ that is orthogonal to $\mathbf{w}_1$; i.e., we set

$$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{w}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1.$$

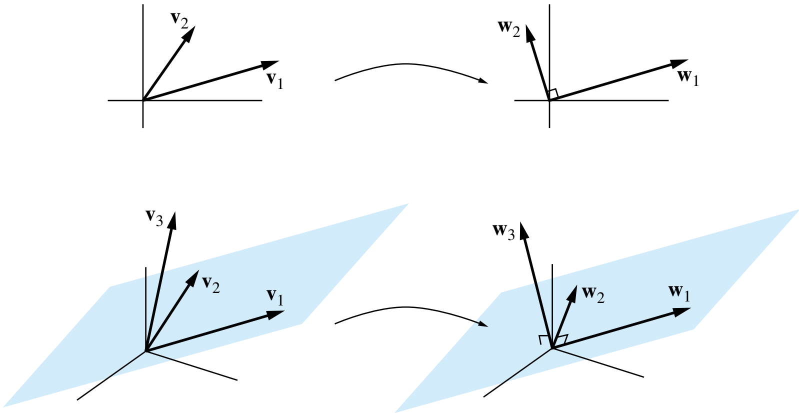

FIGURE 2.2

Then, by construction, $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\text{Span}\,(\mathbf{w}_1, \mathbf{w}_2) \subset \text{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$. Since $\mathbf{w}_2 \neq \mathbf{0}$ (why?), $\{\mathbf{w}_1, \mathbf{w}_2\}$ must be linearly independent and therefore give a basis for $\text{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$ by Proposition 4.4 of Chapter 3. We continue, replacing $\mathbf{v}_3$ by its part orthogonal to the plane spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$\mathbf{w}_3 = \mathbf{v}_3 - \text{proj}_{\text{Span}\,(\mathbf{w}_1,\mathbf{w}_2)}\mathbf{v}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{w}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{w}_2}\mathbf{v}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2.$$

Note that we are making definite use of Proposition 2.3 here: We must use $\mathbf{w}_1$ and $\mathbf{w}_2$ in the formula here, rather than $\mathbf{v}_1$ and $\mathbf{v}_2$, because the formula (†) requires an orthogonal basis. Once again, we find that $\mathbf{w}_3 \neq \mathbf{0}$ (why?), and so $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ must be linearly independent (since they are nonzero and mutually orthogonal) and, resultingly, an orthogonal basis for $\text{Span}\,(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. The process continues until we have arrived at $\mathbf{v}_k$ and replaced it by

$$\mathbf{w}_k = \mathbf{v}_k - \text{proj}_{\text{Span}\,(\mathbf{w}_1,\ldots,\mathbf{w}_{k-1})}\mathbf{v}_k = \mathbf{v}_k - \frac{\mathbf{v}_k \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_k \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \cdots - \frac{\mathbf{v}_k \cdot \mathbf{w}_{k-1}}{\|\mathbf{w}_{k-1}\|^2}\mathbf{w}_{k-1}.$$

Summarizing, we have the algorithm that goes by the name of the Gram-Schmidt process.

Theorem 2.4 (Gram-Schmidt process). *Given a basis* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *for an inner product space* V*, we obtain an* orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *for* V *as follows:*

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \\
&\ \ \vdots
\end{aligned}$$

and, assuming $\mathbf{w}_1, \ldots, \mathbf{w}_j$ *have been defined,*

$$\begin{aligned}
\mathbf{w}_{j+1} &= \mathbf{v}_{j+1} - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \cdots - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_j}{\|\mathbf{w}_j\|^2}\mathbf{w}_j \\
&\ \ \vdots \\
\mathbf{w}_k &= \mathbf{v}_k - \frac{\mathbf{v}_k \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_k \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - \cdots - \frac{\mathbf{v}_k \cdot \mathbf{w}_{k-1}}{\|\mathbf{w}_{k-1}\|^2}\mathbf{w}_{k-1}.
\end{aligned}$$

If we so desire, we can arrange for an orthonormal basis *by dividing each of* $\mathbf{w}_1, \ldots, \mathbf{w}_k$ *by its respective length:*

$$\mathbf{q}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \quad \mathbf{q}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \quad \ldots, \quad \mathbf{q}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|}.$$

EXAMPLE 2

Let $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (3, 1, -1, 1)$, and $\mathbf{v}_3 = (1, 1, 3, 3)$. We want to use the Gram-Schmidt process to give an orthogonal basis for $V = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subset \mathbb{R}^4$. We take

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 = (1, 1, 1, 1);\\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = (3, 1, -1, 1) - \frac{(3, 1, -1, 1) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2}(1, 1, 1, 1)\\
&= (3, 1, -1, 1) - \frac{4}{4}(1, 1, 1, 1) = (2, 0, -2, 0);\\
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2\\
&= (1, 1, 3, 3) - \frac{(1, 1, 3, 3) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2}(1, 1, 1, 1)\\
&\qquad - \frac{(1, 1, 3, 3) \cdot (2, 0, -2, 0)}{\|(2, 0, -2, 0)\|^2}(2, 0, -2, 0)\\
&= (1, 1, 3, 3) - \frac{8}{4}(1, 1, 1, 1) - \frac{-4}{8}(2, 0, -2, 0) = (0, -1, 0, 1).
\end{aligned}$$

And if we desire an *orthonormal* basis, then we take

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{2}(1, 1, 1, 1),\\
\mathbf{q}_2 &= \frac{1}{\sqrt{2}}(1, 0, -1, 0),\\
\mathbf{q}_3 &= \frac{1}{\sqrt{2}}(0, -1, 0, 1).
\end{aligned}$$

It's always a good idea to check that the vectors form an orthogonal (or orthonormal) set, and it's easy—with these numbers—to do so. ▲

2.1 The *QR* Decomposition

The Gram-Schmidt process gives rise in an obvious way to another matrix decomposition that is useful in numerical computation. Just as the reduction to echelon form led to the LU decomposition, the Gram-Schmidt process gives us what is usually called the QR decomposition. We start with a matrix A whose columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a linearly independent set. Let $\mathbf{q}_1, \dots, \mathbf{q}_n$ be the orthonormal basis for $\mathbf{C}(A)$ obtained by applying the Gram-Schmidt process to $\mathbf{v}_1, \dots, \mathbf{v}_n$. Notice that for $j = 1, \dots, n$, the vector $\mathbf{v}_j$ can be written as a linear combination of $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_j$ (why?). Thus, if we let Q be the $m \times n$ matrix with columns $\mathbf{q}_1, \dots, \mathbf{q}_n$, we see that there is an upper triangular $n \times n$ matrix R so that $A = QR$. Moreover, R is nonsingular, since its diagonal entries are nonzero.

EXAMPLE 3

Revisiting Example 2 above, let

$$A = \begin{bmatrix} 1 & 3 & 1\\ 1 & 1 & 1\\ 1 & -1 & 3\\ 1 & 1 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 2 & 2 & 4\\ 0 & 2\sqrt{2} & -\sqrt{2}\\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Then, as the reader can check, we have $A = QR$. The entries of R can be computed by keeping track of the arithmetic during the Gram-Schmidt process, or, more easily, by noting that $r_{ij} = \mathbf{q}_i \cdot \mathbf{v}_j$. (Note, as a check, that $r_{ij} = 0$ whenever $i > j$.) ▲

Remark. The fact that the columns of the $m \times n$ matrix Q form an orthonormal set can be restated in matrix form as $Q^{\mathsf{T}}Q = I_n$. When A is square, so is the matrix Q. An $n \times n$ matrix Q is called *orthogonal* if its column vectors form an *orthonormal* set,[4] i.e., if $Q^{\mathsf{T}}Q = I_n$. Orthogonal matrices will have a geometric role to play soon and will reappear seriously in Chapter 6.

Remark. Suppose we have calculated the QR decomposition of an $m \times n$ matrix A whose columns form a linearly independent set. How does this help us deal with the normal equations and the projection formula? Recall that the normal equations are

$$(A^{\mathsf{T}}A)\overline{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b};$$

substituting $A = QR$, we have

$$R^{\mathsf{T}}(Q^{\mathsf{T}}Q)R\overline{\mathbf{x}} = R^{\mathsf{T}}Q^{\mathsf{T}}\mathbf{b}, \quad \text{or}$$
$$R\overline{\mathbf{x}} = Q^{\mathsf{T}}\mathbf{b}.$$

(Here we've used the facts that $Q^{\mathsf{T}}Q = I_n$ and that R is nonsingular, and hence R^{T} is nonsingular.) In particular, using the formula

$$\overline{\mathbf{x}} = R^{-1}Q^{\mathsf{T}}\mathbf{b}$$

to solve for the least squares solution is quite effective for computer work. Finally, the projection matrix $(*)$ on p. 194 can be rewritten

$$P_V = A(A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}} = (QR)(R^{\mathsf{T}}R)^{-1}(R^{\mathsf{T}}Q^{\mathsf{T}}) = (QR)(R^{-1})(R^{\mathsf{T}})^{-1}(R^{\mathsf{T}})Q^{\mathsf{T}} = QQ^{\mathsf{T}},$$

which says that projecting onto the subspace $\mathbf{C}(A)$ is the same as summing the projections onto the orthonormal basis vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$. This is exactly what Proposition 2.3 told us.

Exercises 4.2

1. Redo Exercise 4.1.4 by finding an orthogonal basis for V.
2. Execute the Gram-Schmidt process in each case to give an orthonormal basis for the subspace spanned by the given vectors.
   a. (1, 0, 0), (2, 1, 0), (3, 2, 1)
   b. (1, 1, 1), (0, 1, 1), (0, 0, 1)
   *c. (1, 0, 1, 0), (2, 1, 0, 1), (0, 1, 2, −3)
   d. (−1, 2, 0, 2), (2, −4, 1, −4), (−1, 3, 1, 1)
3. Let $V = \text{Span}\big((2, 1, 0, -2), (3, 3, 1, 0)\big) \subset \mathbb{R}^4$.
   a. Find an orthogonal basis for V.
   b. Use your answer to part *a* to find the projection of $\mathbf{b} = (0, 4, -4, -7)$ onto V.
   c. Use your answer to part *a* to find the projection matrix P_V.

[4] The terminology is confusing and unfortunate, but history has set it in stone. Matrices with orthogonal column vectors of nonunit length don't have a special name.

*4. Let $V = \text{Span}\big((1, 3, 1, 1), (1, 1, 1, 1), (-1, 5, 2, 2)\big) \subset \mathbb{R}^4$.
   a. Find an orthogonal basis for V.
   b. Use your answer to part *a* to find $\text{proj}_V(4, -1, 5, 1)$.
   c. Use your answer to part *a* to find the projection matrix P_V.

*5. Let $V = \text{Span}\big((1, -1, 0, 2), (1, 0, 1, 1)\big) \subset \mathbb{R}^4$, and let $\mathbf{b} = (1, -3, 1, 1)$.
   a. Find an orthogonal basis for V.
   b. Use your answer to part *a* to find $\mathbf{p} = \text{proj}_V \mathbf{b}$.
   c. Letting

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix},$$

   use your answer to part *b* to give the least squares solution of $A\mathbf{x} = \mathbf{b}$.

6. Let $V = \text{Span}\big((1, 0, 1, 1), (0, 1, 0, 1)\big) \subset \mathbb{R}^4$.
   a. Give an orthogonal basis for V.
   b. Give a basis for the orthogonal complement of V.
   c. Given a general vector $\mathbf{x} \in \mathbb{R}^4$, find $\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$ so that $\mathbf{x} = \mathbf{v} + \mathbf{w}$.

7. According to Proposition 4.10 of Chapter 3, if A is an $m \times n$ matrix, then for each $\mathbf{b} \in \mathbf{C}(A)$, there is a unique $\mathbf{x} \in \mathbf{R}(A)$ with $A\mathbf{x} = \mathbf{b}$. In each case, give a formula for that $\mathbf{x}$. (*Hint:* It may help to remember that all solutions of $A\mathbf{x} = \mathbf{b}$ have the same projection onto $\mathbf{R}(A)$.)

   a. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

   *b. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

   c. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -5 \end{bmatrix}$

   d. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -5 \\ 2 & 2 & 4 & -4 \end{bmatrix}$

*8. Give the QR decomposition of the following matrices.

   a. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

   b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

9. By finding the QR decomposition of the appropriate matrix, redo
   a. Exercise 4.1.6 b. Exercise 4.1.7 c. Exercise 4.1.8

10. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are orthonormal and that for every $\mathbf{x} \in \mathbb{R}^n$ we have

$$\|\mathbf{x}\|^2 = (\mathbf{x} \cdot \mathbf{v}_1)^2 + \cdots + (\mathbf{x} \cdot \mathbf{v}_k)^2.$$

   Prove that $k = n$ and deduce that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthonormal basis for $\mathbb{R}^n$. (*Hint:* If $k < n$, choose $\mathbf{x} \neq \mathbf{0}$ orthogonal to $\text{Span}\,(\mathbf{v}_1, \dots, \mathbf{v}_k)$.)

♯11. Let A be an $n \times n$ matrix and, as usual, let $\mathbf{a}_1, \dots, \mathbf{a}_n$ denote its column vectors.
   a. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_n$ form an *orthonormal* set. Show that $A^{-1} = A^\mathsf{T}$.
   *b. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_n$ form an *orthogonal* set and each is nonzero. Find the appropriate formula for A^{-1}.

12. Using the inner product defined in Example 10(c) of Chapter 3, Section 6, find an orthogonal basis for the given subspace V and use your answer to find the projection of f onto V.
*a. $V = \mathcal{P}_1 \subset \mathcal{C}^0([-1, 1])$, $f(t) = t^2 - t + 1$
b. $V = \mathcal{P}_1 \subset \mathcal{C}^0([0, 1])$, $f(t) = t^2 + t - 1$
c. $V = \mathcal{P}_2 \subset \mathcal{C}^0([-1, 1])$, $f(t) = t^3$
*d. $V = \text{Span}\,(1, \cos t, \sin t) \subset \mathcal{C}^0([-\pi, \pi])$, $f(t) = t$
e. $V = \text{Span}\,(1, \cos t, \sin t) \subset \mathcal{C}^0([-\pi, \pi])$, $f(t) = |t|$

3 The Matrix of a Linear Transformation and the Change-of-Basis Formula

We now elaborate on the discussion of linear transformations initiated in Section 2 of Chapter 2. As we saw in Section 1, the projection of vectors in $\mathbb{R}^n$ onto a subspace $V \subset \mathbb{R}^n$ is a linear transformation (see Exercise 4.1.13), just like the projections, reflections, and rotations in $\mathbb{R}^2$ that we studied in Section 2 of Chapter 2. Other examples of linear transformations include reflections across planes in $\mathbb{R}^3$, rotations in $\mathbb{R}^3$, and even differentiation and integration of polynomials. We will save this last example for the next section; here we will deal only with functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

We recall the following definition.

Definition. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* (or *linear map*) if it satisfies

(i) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(ii) $T(c\mathbf{x}) = c\,T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and scalars c.

If we think visually of T as mapping $\mathbb{R}^n$ to $\mathbb{R}^m$, then we have a diagram like Figure 3.1. We also see that, because of linearity, the behavior of T on *all* of Span $(\mathbf{x}, \mathbf{y})$ is completely determined by $T(\mathbf{x})$ and $T(\mathbf{y})$.

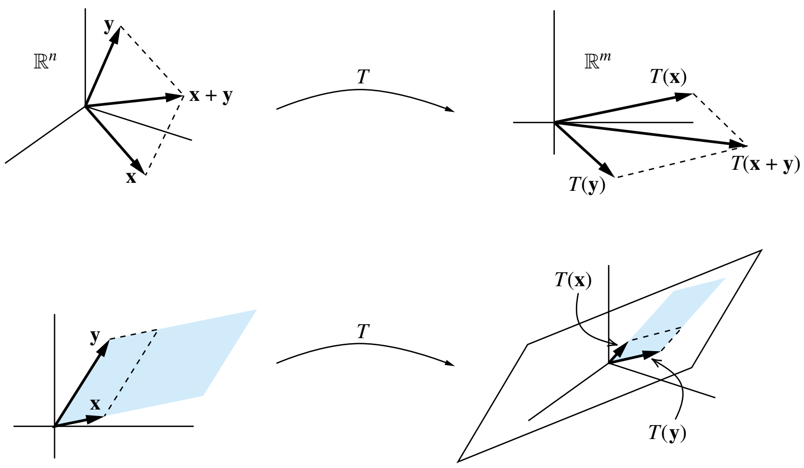

FIGURE 3.1

EXAMPLE 1

We begin with a few examples of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(a) If A is an $m \times n$ matrix, then the map $\mu_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, as we saw in Exercise 1.4.13.

(b) If $V \subset \mathbb{R}^n$ is a subspace, then, as we saw in Section 1, the projection map proj_V is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(c) If $V \subset \mathbb{R}^n$ is a subspace and $\mathbf{x} \in \mathbb{R}^n$, we can define the *reflection* of $\mathbf{x}$ *across* V by the formula

$$R_V(\mathbf{x}) = \text{proj}_V\mathbf{x} - \text{proj}_{V^\perp}\mathbf{x}.$$

Since both proj_V and $\text{proj}_{V^\perp}$ are linear maps, it will follow easily that R_V is, as well: First, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} R_V(\mathbf{x}+\mathbf{y}) &= \text{proj}_V(\mathbf{x}+\mathbf{y}) - \text{proj}_{V^\perp}(\mathbf{x}+\mathbf{y}) \\ &= \big(\text{proj}_V\mathbf{x} + \text{proj}_V\mathbf{y}\big) - \big(\text{proj}_{V^\perp}\mathbf{x} + \text{proj}_{V^\perp}\mathbf{y}\big) \\ &= \big(\text{proj}_V\mathbf{x} - \text{proj}_{V^\perp}\mathbf{x}\big) + \big(\text{proj}_V\mathbf{y} - \text{proj}_{V^\perp}\mathbf{y}\big) \\ &= R_V(\mathbf{x}) + R_V(\mathbf{y}). \end{aligned}$$

Next, for all $\mathbf{x} \in \mathbb{R}^n$ and scalars c, we have

$$\begin{aligned} R_V(c\mathbf{x}) &= \text{proj}_V(c\mathbf{x}) - \text{proj}_{V^\perp}(c\mathbf{x}) \\ &= c\,\text{proj}_V\mathbf{x} - c\,\text{proj}_{V^\perp}\mathbf{x} = c\big(\text{proj}_V\mathbf{x} - \text{proj}_{V^\perp}\mathbf{x}\big) \\ &= cR_V(\mathbf{x}), \end{aligned}$$

as required. ▲

As we saw in Section 2 of Chapter 2, if $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then we can find a matrix A, the so-called standard matrix of A, so that $T = \mu_A$: The j^{th} column of A is given by $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} standard basis vector. We summarize this in the following proposition.

Proposition 3.1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Let A be the matrix whose* column vectors *are the vectors $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n) \in \mathbb{R}^m$ (that is, the coordinate vectors of $T(\mathbf{e}_j)$ with respect to the standard basis of $\mathbb{R}^m$):*

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}.$$

Then $T = \mu_A$ and we call A the standard matrix *for T. We will denote this matrix by $[T]_{\text{stand}}$.*

Proof. This follows immediately from the linearity properties of T. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) + \cdots + T(x_n\mathbf{e}_n) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

□

EXAMPLE 2

The most basic example of a linear map is the following. Fix $\mathbf{a} \in \mathbb{R}^n$, and define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$. By Proposition 2.1 of Chapter 1, we have

$$T(\mathbf{u} + \mathbf{v}) = \mathbf{a} \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{a} \cdot \mathbf{u}) + (\mathbf{a} \cdot \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \text{and}$$
$$T(c\mathbf{v}) = \mathbf{a} \cdot (c\mathbf{v}) = c(\mathbf{a} \cdot \mathbf{v}) = c\,T(\mathbf{v}),$$

as required. Moreover, it is easy to see that

$$\text{if} \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \text{then} \quad [T]_{\text{stand}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \mathbf{a}^{\mathsf{T}}$$

is the standard matrix for T, as we might expect. ▲

EXAMPLE 3

Let $V \subset \mathbb{R}^3$ be the plane $x_3 = 0$ and consider the linear transformation $\text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$. We now use Proposition 3.1 to find its standard matrix. Since $\text{proj}_V(\mathbf{e}_1) = \mathbf{e}_1$, $\text{proj}_V(\mathbf{e}_2) = \mathbf{e}_2$, and $\text{proj}_V(\mathbf{e}_3) = \mathbf{0}$, we have

$$[\text{proj}_V]_{\text{stand}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, consider the reflection R_V across the plane V. Since $R_V(\mathbf{e}_1) = \mathbf{e}_1$, $R_V(\mathbf{e}_2) = \mathbf{e}_2$, and $R_V(\mathbf{e}_3) = -\mathbf{e}_3$, the standard matrix for R_V is

$$[R_V]_{\text{stand}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

▲

EXAMPLE 4

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating an angle θ counterclockwise around the x_3-axis, as viewed from high above the x_1x_2-plane. Then we have (see Example 6 in Section 2 of Chapter 2)

$$T(\mathbf{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_3) = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$[T]_{\text{stand}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

▲

EXAMPLE 5

Now let $V \subset \mathbb{R}^3$ be the plane spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Let's compute the standard matrix for proj_V using Proposition 3.1. Noting that $\{\mathbf{v}_1, \mathbf{v}_2\}$ gives an orthogonal basis for V, we can use Proposition 2.3 to compute

$$\begin{aligned}
\text{proj}_V(\mathbf{e}_1) &= \text{proj}_{\mathbf{v}_1}(\mathbf{e}_1) + \text{proj}_{\mathbf{v}_2}(\mathbf{e}_1) \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \\
\text{proj}_V(\mathbf{e}_2) &= \text{proj}_{\mathbf{v}_1}(\mathbf{e}_2) + \text{proj}_{\mathbf{v}_2}(\mathbf{e}_2) \\
&= \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \\
\text{proj}_V(\mathbf{e}_3) &= \text{proj}_{\mathbf{v}_1}(\mathbf{e}_3) + \text{proj}_{\mathbf{v}_2}(\mathbf{e}_3) \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{-1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}.
\end{aligned}$$

Thus, the standard matrix for proj_V is given by

$$[\text{proj}_V]_{\text{stand}} = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}.$$

Of course, in general, $[\text{proj}_V]_{\text{stand}}$ is nothing but the projection matrix P_V we computed in Section 1. ▲

EXAMPLE 6

Using the same plane V as in Example 5, we can compute the standard matrix for $\text{proj}_{V^\perp}$ because we know that $\text{proj}_V + \text{proj}_{V^\perp}$ is the identity map. Thus, the standard matrix for $\text{proj}_{V^\perp}$ is given by

$$[\text{proj}_{V^\perp}]_{\text{stand}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$

To double-check our work, note that the column space (and row space, too) is spanned by $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, which is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$ and therefore spans $V^\perp$.

Similarly, we can compute the standard matrix for reflection across V by using the definition:

$$\begin{aligned}[R_V]_{\text{stand}} &= [\text{proj}_V]_{\text{stand}} - [\text{proj}_{V^\perp}]_{\text{stand}} \\ &= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & 4 & 2 \\ 4 & -2 & -4 \\ 2 & -4 & 4 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{bmatrix}.\end{aligned}$$

▲

We see in these examples that the standard matrix for a projection or reflection may disguise the geometry of the actual linear transformation. When $\mathbf{x} \in V$ and $\mathbf{y} \in V^\perp$, we know that $\text{proj}_V(\mathbf{x}) = \mathbf{x}$ and $\text{proj}_V(\mathbf{y}) = \mathbf{0}$; thus, in Example 5, since $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\mathbf{v}_3 \in V^\perp$, we have

$$\begin{aligned}\text{proj}_V(\mathbf{v}_1) &= \mathbf{v}_1, \\ \text{proj}_V(\mathbf{v}_2) &= \mathbf{v}_2, \\ \text{proj}_V(\mathbf{v}_3) &= \mathbf{0}.\end{aligned}$$

Now, since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal nonzero vectors, they form a basis for $\mathbb{R}^3$. Thus, given any $\mathbf{x} \in \mathbb{R}^3$, we can write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for appropriate scalars c_1, c_2, and c_3. Since proj_V is a linear transformation, we therefore have

$$\begin{aligned}\text{proj}_V(\mathbf{x}) &= \text{proj}_V(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\ &= c_1\text{proj}_V(\mathbf{v}_1) + c_2\text{proj}_V(\mathbf{v}_2) + c_3\text{proj}_V(\mathbf{v}_3) \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2.\end{aligned}$$

Similarly, when $\mathbf{x} \in V$ and $\mathbf{y} \in V^\perp$, we know that $R_V(\mathbf{x}) = \mathbf{x}$ and $R_V(\mathbf{y}) = -\mathbf{y}$, so

$$R_V(\mathbf{x}) = R_V(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - c_3\mathbf{v}_3.$$

These examples lead us to make the following definition.

Definition. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for $\mathbb{R}^n$. For each $j = 1, \ldots, n$, let $a_{1j}, a_{2j}, \ldots, a_{nj}$ denote the coordinates of $T(\mathbf{v}_j)$ with respect to the basis $\mathcal{B}$, i.e.,

$$T(\mathbf{v}_j) = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \cdots + a_{nj}\mathbf{v}_n, \qquad j = 1, \ldots, n.$$

Then we define $A = [a_{ij}]$ to be the *matrix for T with respect to the basis $\mathcal{B}$*. We denote this matrix by $[T]_{\mathcal{B}}$.

It is important to remember that the coefficients of $T(\mathbf{v}_j)$ will be entered as the j^{th} *column* of the matrix $[T]_{\mathcal{B}}$. To write this formally, given a vector $\mathbf{x} \in \mathbb{R}^n$, we let $C_{\mathcal{B}}(\mathbf{x})$ denote the column vector whose entries are the coordinates of $\mathbf{x}$ with respect to the basis

$\mathcal{B}$. That is, if

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \quad \text{then} \quad C_{\mathcal{B}}(\mathbf{x}) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

With this notation, we now can write

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ C_{\mathcal{B}}(T(\mathbf{v}_1)) & C_{\mathcal{B}}(T(\mathbf{v}_2)) & \cdots & C_{\mathcal{B}}(T(\mathbf{v}_n)) \\ | & | & & | \end{bmatrix}.$$

Remark. If we denote the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ by $\mathcal{E}$, we have $[T]_{\mathcal{E}} = [T]_{\text{stand}}$.

EXAMPLE 7

Returning to Examples 5 and 6, when we take $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

and $V = \text{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$, we have

$$[\text{proj}_V]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\text{proj}_{V^\perp}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{and} \quad [R_V]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In fact, it doesn't matter what the particular vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ happen to be. So long as $V = \text{Span}\,(\mathbf{v}_1, \mathbf{v}_2)$ and $V^\perp = \text{Span}\,(\mathbf{v}_3)$, we will obtain these matrices with respect to the basis $\mathcal{B}$. Indeed, these are the matrices we obtained for projection onto and reflection across the standard plane $x_3 = 0$ in Example 3. ▲

EXAMPLE 8

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by multiplying by

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

It is rather difficult to understand this mapping until we discover that if we take

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

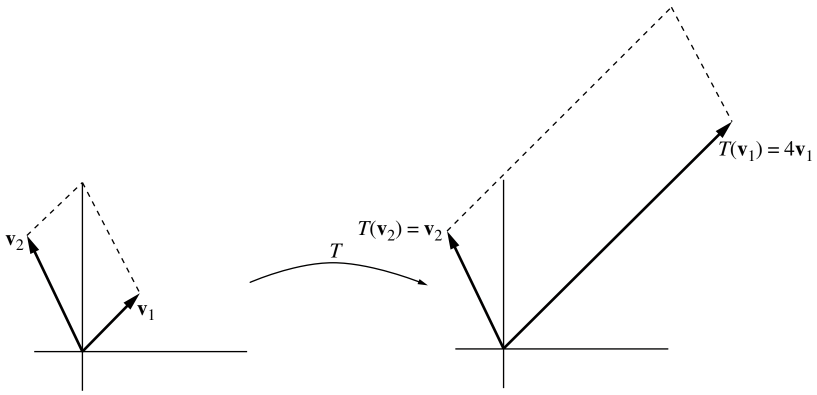

FIGURE 3.2

then $T(\mathbf{v}_1) = 4\mathbf{v}_1$ and $T(\mathbf{v}_2) = \mathbf{v}_2$, as shown in Figure 3.2, so that the matrix for T with respect to the ordered basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is the diagonal matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now it is rather straightforward to picture the linear transformation: It stretches the $\mathbf{v}_1$-axis by a factor of 4 and leaves the $\mathbf{v}_2$-axis unchanged. Because we can "pave" the plane by parallelograms formed by $\mathbf{v}_1$ and $\mathbf{v}_2$, we are able to describe the effects of T quite explicitly. The curious reader will learn how we stumbled upon $\mathbf{v}_1$ and $\mathbf{v}_2$ by reading Chapter 6.

At first it might seem confusing that the same transformation is represented by more than one matrix. Indeed, each choice of basis gives a different matrix for the transformation T. How are these matrices related to each other? The answer is found by looking at the matrix P with column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:[5]

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Since $T(\mathbf{v}_1) = 4\mathbf{v}_1$ and $T(\mathbf{v}_2) = \mathbf{v}_2$, we observe that the first column of AP, $A\mathbf{v}_1$, is $4\mathbf{v}_1 = P\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and the second column of AP, $A\mathbf{v}_2$, is $\mathbf{v}_2 = P\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore,

$$AP = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = P[T]_{\mathcal{B}}.$$

This might be rewritten as $[T]_{\mathcal{B}} = P^{-1}AP$, in the form that will occupy our attention for the rest of this section.

It would have been a more honest exercise here to start with the geometric description of T, i.e., its action on the basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and try to find the standard matrix for T. As the reader can check, we have

$$\begin{aligned} \mathbf{e}_1 &= \tfrac{2}{3}\mathbf{v}_1 - \tfrac{1}{3}\mathbf{v}_2 \\ \mathbf{e}_2 &= \tfrac{1}{3}\mathbf{v}_1 + \tfrac{1}{3}\mathbf{v}_2, \end{aligned}$$

and so we compute that

[5] In the ensuing discussion, and on into Chapter 6, the matrix P has nothing to do with projection matrices.

$$T(\mathbf{e}_1) = \tfrac{2}{3}T(\mathbf{v}_1) - \tfrac{1}{3}T(\mathbf{v}_2) = \tfrac{8}{3}\mathbf{v}_1 - \tfrac{1}{3}\mathbf{v}_2$$
$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and}$$
$$T(\mathbf{e}_2) = \tfrac{1}{3}T(\mathbf{v}_1) + \tfrac{1}{3}T(\mathbf{v}_2) = \tfrac{4}{3}\mathbf{v}_1 + \tfrac{1}{3}\mathbf{v}_2$$
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

What a relief! In matrix form, this would, of course, be the equation $A = [T]_{\text{stand}} = P[T]_{\mathcal{B}}P^{-1}$. ▲

Following this example, we now state the main result of this section.

Proposition 3.2 (Change-of-Basis Formula, Take 1). *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix $[T]_{\text{stand}}$. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis for $\mathbb{R}^n$ and let $[T]_{\mathcal{B}}$ be the matrix for T with respect to $\mathcal{B}$. Let P be the $n \times n$ matrix whose columns are given by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then we have*

$$[T]_{\text{stand}}P = P[T]_{\mathcal{B}}.$$

Remark. Since P is invertible (why?), this can be written as $[T]_{\text{stand}} = P[T]_{\mathcal{B}}P^{-1}$ or as $[T]_{\mathcal{B}} = P^{-1}[T]_{\text{stand}}P$, providing us formulas to find $[T]_{\text{stand}}$ from $[T]_{\mathcal{B}}$, and vice versa. Since these formulas tell us how to change the matrix from one basis to another, we call them *change-of-basis formulas*. We call P the *change-of-basis matrix from the standard basis to the basis $\mathcal{B}$*. Note that, as a linear transformation,

$$\mu_P(\mathbf{e}_j) = \mathbf{v}_j, \quad j = 1, \ldots, n;$$

that is, it maps the j^{th} standard basis vector to the j^{th} element of the new ordered basis.

Proof. The j^{th} column of P is the vector $\mathbf{v}_j$ (more specifically, its coordinate vector $C_{\mathcal{E}}(\mathbf{v}_j)$ with respect to the standard basis). Therefore, the j^{th} column vector of the matrix product $[T]_{\text{stand}}P$ is the standard coordinate vector of $T(\mathbf{v}_j)$. On the other hand, the j^{th} column of $[T]_{\mathcal{B}}$ is the coordinate vector, $C_{\mathcal{B}}(T(\mathbf{v}_j))$, of $T(\mathbf{v}_j)$ with respect to the basis $\mathcal{B}$. That is, if the j^{th} column of $[T]_{\mathcal{B}}$ is $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ then $T(\mathbf{v}_j) = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$. But we also know that

$$P\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

That is, the j^{th} column of $P[T]_{\mathcal{B}}$ is exactly the linear combination of the columns of P needed to give the standard coordinate vector of $T(\mathbf{v}_j)$. □

EXAMPLE 9

We want to use the change-of-basis formula, Proposition 3.2, and the result of Example 7 to recover the results of Examples 5 and 6. Starting with the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix},$$

we note that $V = \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2)$ and so

$$[\operatorname{proj}_V]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To get the standard matrix for proj_V, we take

$$P = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

and compute that

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}.$$

(Exercise 4.2.11 may be helpful here, but, as a last resort, there's always Gaussian elimination.) Then we have

$$\begin{aligned}
[\operatorname{proj}_V]_{\text{stand}} = P[\operatorname{proj}_V]_{\mathcal{B}}P^{-1} &= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix},
\end{aligned}$$

as before. Similarly,

$$\begin{aligned}
[R_V]_{\text{stand}} = P[R_V]_{\mathcal{B}}P^{-1} &= \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{bmatrix},
\end{aligned}$$

just as we obtained previously. ▲

EXAMPLE 10

Armed with the change-of-basis formula, we can now study rotations in $\mathbb{R}^3$. Consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by rotating an angle $2\pi/3$ about the line spanned by $(1, -1, 1)$. (The angle is measured counterclockwise, looking down from a vantage point on the "positive side" of this line; see Figure 3.3.) To find the standard matrix for T, the key is to choose a convenient basis $\mathcal{B}$ adapted to the geometry of the problem. We choose

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

along the axis of rotation and $\mathbf{v}_1, \mathbf{v}_2$ to be an orthonormal basis for the plane orthogonal to that axis; for example,

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

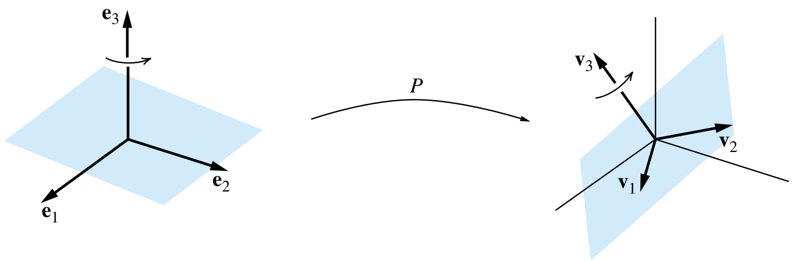

FIGURE 3.3

(How did we arrive at these?) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ rotate an angle $2\pi/3$, we have (see Example 6 in Section 2 of Chapter 2)

$$\begin{aligned} T(\mathbf{v}_1) &= -\tfrac{1}{2}\mathbf{v}_1 + \tfrac{\sqrt{3}}{2}\mathbf{v}_2 \\ T(\mathbf{v}_2) &= -\tfrac{\sqrt{3}}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_2 \\ T(\mathbf{v}_3) &= \mathbf{v}_3. \end{aligned}$$

(Now the alert reader should figure out why we chose $\mathbf{v}_1, \mathbf{v}_2$ to be orthonormal. We also want $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to form a "right-handed system" so that we're turning in the correct direction. But there's no need to worry about the length of $\mathbf{v}_3$.) Thus, we have

$$[T]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, the change-of-basis matrix from the standard basis to the new basis $\mathcal{B}$ is

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -1 \\ 0 & \frac{2}{\sqrt{6}} & 1 \end{bmatrix},$$

whose inverse matrix is

$$P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Once again, we solve for

$$[T]_{\text{stand}} = P[T]_{\mathcal{B}}P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -1 \\ 0 & \frac{2}{\sqrt{6}} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

amazingly enough. In hindsight, then, we should be able to see the effect of T on the standard basis vectors quite plainly. Can you? ▲

Remark. Suppose we first rotate $\pi/2$ about the x_3-axis and then rotate $\pi/2$ about the x_1-axis. We leave it to the reader to check, in Exercise 2, that the result is the linear transformation whose matrix we just calculated. This raises a fascinating question: Is the composition of rotations always again a rotation? (See Exercise 6.2.16.) If so, is there a way of predicting the ultimate axis and angle?

EXAMPLE 11

Consider the ordered basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by

$$T(\mathbf{v}_1) = \mathbf{v}_2 - \mathbf{v}_3, \quad T(\mathbf{v}_2) = -\mathbf{v}_2, \quad \text{and} \quad T(\mathbf{v}_3) = \mathbf{v}_1 + \mathbf{v}_2.$$

Then the matrix $[T]_{\mathcal{B}}$ for T with respect to the basis $\mathcal{B}$ is given by

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

To compute the standard matrix for T, we take

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & -2 \end{bmatrix}$$

and calculate

$$P^{-1} = \begin{bmatrix} 4 & 5 & -3 \\ -2 & -3 & 2 \\ -1 & -2 & 1 \end{bmatrix}.$$

Then

$$[T]_{\text{stand}} = P[T]_{\mathcal{B}}P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & -3 \\ -2 & -3 & 2 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 13 & 16 & -10 \\ 22 & 26 & -17 \end{bmatrix}.$$

▲

We end this section with a definition and a few comments.

Definition. Let A and B be $n \times n$ matrices. We say B is *similar* to A if there is an invertible matrix P such that $B = P^{-1}AP$.

That is, two square matrices are similar if they are the matrices for some linear transformation with respect to different ordered bases.

We say a linear transformation $T: V \to V$ is *diagonalizable* if there is some basis $\mathcal{B}$ for V such that the matrix for T with respect to that basis is diagonal. Similarly,[6] we say a square matrix is *diagonalizable* if it is similar to a diagonal matrix. In Chapter 6 we will see the power of diagonalizing matrices to solve a wide variety of problems, and we will learn some convenient criteria for matrices to be diagonalizable.

EXAMPLE 12

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

To decide whether B is similar to A, we try to find an invertible matrix

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that $B = P^{-1}AP$, or, equivalently, $PB = AP$. Calculating, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2a & 3b \\ 2c & 3d \end{bmatrix} = \begin{bmatrix} 2a + c & 2b + d \\ 3c & 3d \end{bmatrix}$$

[6]One of the authors apologizes for the atrocious pun; the other didn't even notice.

if and only if $c = 0$ and $b = d$. Setting $a = b = d = 1$ and $c = 0$, we can check that, indeed, with

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

we have $PB = AP$, and so, since P is invertible, $B = P^{-1}AP$. In particular, we infer that A is diagonalizable. ▲

Exercises 4.3

The first four exercises are meant to be a review of material from Section 2 of Chapter 2: Just find the standard matrices by determining where each of the standard basis vectors is mapped.

1. Calculate the standard matrix for each of the following linear transformations T:

*a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating $-\pi/4$ about the origin and then reflecting across the line $x_1 - x_2 = 0$

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating $\pi/2$ about the x_1-axis (as viewed from the positive side) and then reflecting across the plane $x_2 = 0$

c. $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating $-\pi/2$ about the x_1-axis (as viewed from the positive side) and then rotating $\pi/2$ about the x_3-axis

2. Check the result claimed in the remark on p. 218.

3. Consider the cube with vertices $(\pm 1, \pm 1, \pm 1)$, pictured in Figure 3.4. (Note that the coordinate axes pass through the centers of the various faces.) Give the standard matrices for each of the following symmetries of the cube. Check that each of your matrices is an orthogonal 3×3 matrix.

*a. $90°$ rotation about the x_3-axis (viewed from high above)

b. $180°$ rotation about the line joining $(-1, 0, 1)$ and $(1, 0, -1)$

c. $120°$ rotation about the line joining $(-1, -1, -1)$ and $(1, 1, 1)$ (viewed from high above)

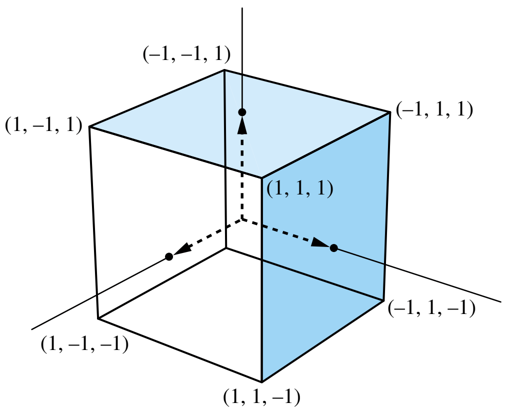

FIGURE 3.4

4. Consider the tetrahedron with vertices $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$, and $(-1, 1, -1)$, pictured in Figure 3.5. Give the standard matrices for each of the following symmetries of the tetrahedron. Check that each of your matrices is an orthogonal 3×3 matrix.

a. $\pm 120°$ rotations about the line joining $(0, 0, 0)$ and the vertex $(1, 1, 1)$
b. $180°$ rotation about the line joining $(0, 0, 1)$ and $(0, 0, -1)$
c. reflection across the plane containing the topmost edge and the midpoint of the opposite edge

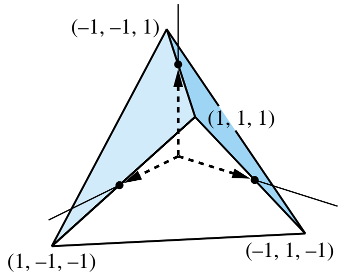

FIGURE 3.5

5. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$.

*a. Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose standard matrix is $[T]_{\text{stand}} = \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$. Find the matrix $[T]_{\mathcal{B}}$.

b. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by

$$\begin{aligned} S(\mathbf{v}_1) &= 2\mathbf{v}_1 + \mathbf{v}_2 \\ S(\mathbf{v}_2) &= -\mathbf{v}_1 + 3\mathbf{v}_2, \end{aligned}$$

then give the standard matrix for S.

6. Derive the result of Exercise 2.2.12 by the change-of-basis formula.

***7.** The standard matrix for a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$. Use the change-of-basis formula to find its matrix with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

8. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Choose a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for V and a basis $\{\mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for $V^\perp$. Then $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms a basis for $\mathbb{R}^n$. Consider the linear transformations proj_V, $\text{proj}_{V^\perp}$, and R_V, all mapping $\mathbb{R}^n$ to $\mathbb{R}^n$, given by projection to V, projection to $V^\perp$, and reflection across V, respectively. Give the matrices for these three linear transformations with respect to the basis $\mathcal{B}$.

9. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflecting across the plane $-x_1 + x_2 + x_3 = 0$.

a. Find an orthogonal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ so that $\mathbf{v}_1, \mathbf{v}_2$ span the plane and $\mathbf{v}_3$ is orthogonal to it.

b. Give the matrix representing T with respect to your basis in part a.

c. Use the change-of-basis theorem to give the standard matrix for T.

10. Redo Exercise 4.1.4 by using the change-of-basis formula.

***11.** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflecting across the plane $x_1 - 2x_2 + 2x_3 = 0$. Use the change-of-basis formula to find its standard matrix.

12. Let $V \subset \mathbb{R}^3$ be the subspace defined by

$$V = \{(x_1, x_2, x_3) : x_1 - x_2 + x_3 = 0\}.$$

Find the standard matrix for each of the following linear transformations:

a. projection on V

b. reflection across V

c. rotation of V through angle $\pi/6$ (as viewed from high above)

13. Let V be the subspace of $\mathbb{R}^3$ spanned by $(1, 0, 1)$ and $(0, 1, 1)$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflecting across V. Find the standard matrix for T.

***14.** Let $V \subset \mathbb{R}^3$ be the plane defined by the equation $2x_1 + x_2 = 0$. Find the standard matrix for R_V.

15. Let the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by rotating an angle $\pi/2$ about the line spanned by $(2, 1, 0)$ (viewed from a vantage point far out the "positive side" of this line) and then reflecting across the plane orthogonal to this line. (See Figure 3.6.) Use the change-of-basis formula to give the standard matrix for T.

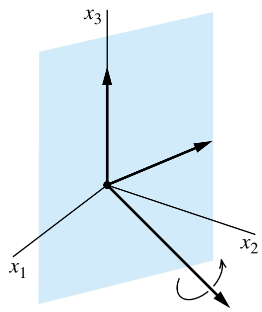

FIGURE 3.6

***16.** Let $V = \text{Span}\big((1, 0, 2, 1), (0, 1, -1, 1)\big) \subset \mathbb{R}^4$. Use the change-of-basis formula to find the standard matrix for $\text{proj}_V\colon \mathbb{R}^4 \to \mathbb{R}^4$.

17. Show (by calculation) that for any real numbers a and b, the matrices $\begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & b \\ 0 & 2 \end{bmatrix}$ are similar. (*Hint:* Remember that when P is invertible, $B = P^{-1}AP \iff PB = AP$.)

18. a. If c is any scalar, show that cI is similar only to itself.

b. Show that $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is similar to $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$.

c. Show that $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not similar to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

d. Show that $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonalizable, i.e., is not similar to any diagonal matrix. (*Hint:* Remember that when P is invertible, $B = P^{-1}AP \iff PB = AP$.)

19. Let $\mathbf{e}_1, \mathbf{e}_2$ denote the standard basis, as usual. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(\mathbf{e}_1) = 8\mathbf{e}_1 - 4\mathbf{e}_2$$
$$T(\mathbf{e}_2) = 9\mathbf{e}_1 - 4\mathbf{e}_2.$$

a. Give the standard matrix for T.

b. Let $\mathbf{v}_1 = 3\mathbf{e}_1 - 2\mathbf{e}_2$ and $\mathbf{v}_2 = -\mathbf{e}_1 + \mathbf{e}_2$. Calculate the matrix for T with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

c. Is T diagonalizable? Give your reasoning. (*Hint:* See part *d* of Exercise 18.)

20. Prove or give a counterexample:

a. If B is similar to A, then B^T is similar to A^T.

b. If B^2 is similar to A^2, then B is similar to A.

c. If B is similar to A and A is nonsingular, then B is nonsingular.

d. If B is similar to A and A is symmetric, then B is symmetric.

e. If B is similar to A, then $\mathbf{N}(B) = \mathbf{N}(A)$.

f. If B is similar to A, then $\text{rank}(B) = \text{rank}(A)$.

21. Show that similarity of matrices is an equivalence relation. That is, verify the following.

a. Reflexivity: Any $n \times n$ matrix A is similar to itself.

b. Symmetry: For any $n \times n$ matrices A and B, if A is similar to B, then B is similar to A.

c. Transitivity: For any $n \times n$ matrices A, B, and C, if A is similar to B and B is similar to C, then A is similar to C.

22. Suppose A and B are $n \times n$ matrices.

a. Show that if either A or B is nonsingular, then AB and BA are similar.

b. Must AB and BA be similar in general?

23. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by rotating some angle θ and about a line spanned by the unit vector $\mathbf{v}$. Let A be the standard matrix for T. Use the change-of-basis formula to prove that A is an orthogonal matrix (i.e., that $A^\mathsf{T} = A^{-1}$). (*Hint:* Choose an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ with $\mathbf{v}_3 = \mathbf{v}$.)

*__24.__ Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose standard matrix is

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} + \frac{\sqrt{6}}{6} & \frac{1}{6} - \frac{\sqrt{6}}{3} \\ \frac{1}{3} - \frac{\sqrt{6}}{6} & \frac{2}{3} & \frac{1}{3} + \frac{\sqrt{6}}{6} \\ \frac{1}{6} + \frac{\sqrt{6}}{3} & \frac{1}{3} - \frac{\sqrt{6}}{6} & \frac{1}{6} \end{bmatrix}.$$

a. Find a nonzero vector $\mathbf{v}_1$ satisfying $A\mathbf{v}_1 = \mathbf{v}_1$. (*Hint:* Proceed as in Exercise 1.4.5.)[7]

b. Find an orthonormal basis $\{\mathbf{v}_2, \mathbf{v}_3\}$ for the plane orthogonal to $\mathbf{v}_1$.

c. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Apply the change-of-basis formula to find $[T]_\mathcal{B}$.

d. Use your answer to part *c* to explain why T is a rotation. (Also see Example 6 in Section 1 of Chapter 7.)

---

[7] This might be a reasonable place to give in and use a computer program like Maple, Mathematica, or MATLAB.

25. a. Show that the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by multiplication by

$$A = \frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

is a rotation. (*Hint:* Proceed as in Exercise 24.)

b. (Calculator suggested) Determine the angle of rotation.

26. *a. Fix $0 < \phi \le \pi/2$ and $0 \le \theta < 2\pi$, and let $\mathbf{a} = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$. Show that the intersection of the circular cylinder $x_1^2 + x_2^2 = 1$ with the plane $\mathbf{a}\cdot\mathbf{x} = 0$ is an ellipse. (*Hint:* Consider the new basis $\mathbf{v}_1 = (\sin\theta, -\cos\theta, 0)$, $\mathbf{v}_2 = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi)$, $\mathbf{v}_3 = \mathbf{a}$.)

b. Describe the projection of the cylindrical region $x_1^2 + x_2^2 = 1$, $-h \le x_3 \le h$ onto the general plane $\mathbf{a}\cdot\mathbf{x} = 0$. (*Hint:* Special cases are the planes $x_3 = 0$ and $x_1 = 0$.)

27. Let the linear map $T:\mathbb{R}^2 \to \mathbb{R}^2$ have standard matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

a. Calculate the matrix for T with respect to the basis $\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

b. Set $\mathbf{y} = C_{\mathcal{B}}(\mathbf{x})$, and calculate $A\mathbf{x}\cdot\mathbf{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2$ in terms of y_1 and y_2.

c. Use the result of part b to sketch the conic section $2x_1^2 - 2x_1x_2 + 2x_2^2 = 3$. (See also Section 4.1 of Chapter 6.)

4 Linear Transformations on Abstract Vector Spaces

In this section we deal with linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ (with m and n different) and, more generally, from one abstract vector space (see Section 6 of Chapter 3) to another. We have the following definition.

Definition. Let V and W be vector spaces. A function $T: V \to W$ is called a *linear transformation* (or *linear map*) if it satisfies

(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$;

(b) $T(c\mathbf{v}) = c\,T(\mathbf{v})$ for all $\mathbf{v} \in V$ and scalars c.

Of course, we can take $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ and all our previous examples would be appropriate as examples here. But the broad scope of linear algebra begins to become apparent when we consider vector spaces such as the set of all matrices or function spaces and consider maps between them. A very important example comes from differential calculus.

EXAMPLE 1

For any interval $\mathcal{I} \subset \mathbb{R}$, define

$$D : \mathcal{C}^1(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I}) \quad \text{by} \quad D(f) = f'.$$

That is, to each continuously differentiable function f, associate its derivative (which then is a continuous function). D satisfies the linearity properties by virtue of the rules of differentiation:

$$\begin{aligned} D(f+g) &= D(f) + D(g) \quad &\text{since} \quad (f+g)' &= f' + g'; \\ D(cf) &= c\,D(f) \quad &\text{since} \quad (cf)' &= cf'. \end{aligned}$$

Although $\mathcal{C}^1(\mathcal{I})$ is infinite-dimensional, we can also think of restricting this linear transformation to smaller (possibly finite-dimensional) subspaces. For example, we can consider $D : \mathcal{P}_k \to \mathcal{P}_{k-1}$ for any positive integer k, since the derivative of a polynomial is a polynomial of one degree less. ▲

EXAMPLE 2

Here are some more examples of linear transformations on abstract vector spaces.

(a) The map $M : \mathcal{C}^0(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I})$ given by $M(f)(t) = tf(t)$

(b) The map $T : \mathcal{C}^0([0, 1]) \to \mathcal{C}^0([0, 1])$ given by $T(f)(t) = \int_0^t f(s)\,ds$

(c) The map $E : \mathcal{C}^0([0, 4]) \to \mathbb{R}^3$ given by $E(f) = \begin{bmatrix} f(0) \\ f(1) \\ f(3) \end{bmatrix}$

(d) The map $T : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ given by $T(A) = BA$, for $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

We leave it to the reader to check that these are all linear transformations. It is also worth thinking about how one might restrict the domains and ranges to various subspaces, e.g., the subspaces of polynomials or polynomials of a certain degree. ▲

Just as the nullspace and column space are crucial tools to understand the linear map $\mu_A : \mathbb{R}^n \to \mathbb{R}^m$ associated to an $m \times n$ matrix A, we define corresponding subspaces for arbitrary linear transformations.

Definition. Let $T : V \to W$ be a linear transformation. We define

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\},$$

called the *kernel of* T, *and*

$$\text{image}\,(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\},$$

called the *image of* T.

We leave it to the reader to check, in Exercise 11, that the kernel of T is a subspace of V and that the image of T is a subspace of W.

EXAMPLE 3

Let's determine the kernel and image of a few linear transformations.

(a) Consider the differentiation map $D\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $D(f) = f'$. Then the constant function 1 gives a basis for $\ker(D)$ (see Exercise 6), and $\text{image}\,(D) = \mathcal{P}_2$, since, given $g(t) = a + bt + ct^2 \in \mathcal{P}_2$, we set $f(t) = at + \frac{1}{2}bt^2 + \frac{1}{3}ct^3$ and $D(f) = g$.

(b) Consider the linear map M defined in Example 2(a) with the interval $\mathcal{I} = [1, 2]$. If $f \in \ker(M) = \{f \in \mathcal{C}^0(\mathcal{I}) : tf(t) = 0 \text{ for all } t \in \mathcal{I}\}$, then we must have $f(t) = 0$ for all $t \in \mathcal{I}$. Given any continuous function g, we can take $f(t) = g(t)/t$ and this too will be continuous; since $M(f) = g$, we see that $\text{image}\,(M) = \mathcal{C}^0(\mathcal{I})$. We ask the reader to explore, in Exercise 15, what happens for an interval $\mathcal{I}$ containing 0.

(c) Consider the linear map $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ defined by $T(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$. Then $\ker(T)$ consists of all quadratic polynomials with roots at 0 and 1, i.e., all constant multiples of $f(t) = t(t-1)$, so f gives a basis for $\ker(T)$. On the other hand, given any $(a, b) \in \mathbb{R}^2$, we can set $f(t) = a + (b-a)t$ and $T(f) = (a, b)$.

(d) Modifying the preceding example slightly, consider the linear map $S\colon \mathcal{P}_2 \to \mathbb{R}^3$ defined by $S(f) = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$. Now $\ker(S) = \{\mathbf{0}\}$ since a nonzero quadratic polynomial can have only two roots. On the other hand, it follows from the Lagrange Interpolation Formula, Theorem 6.4 of Chapter 3, that for every $(a, b, c) \in \mathbb{R}^3$, there is a polynomial $f \in \mathcal{P}_2$ with $S(f) = (a, b, c)$. Explicitly, we take

$$\begin{aligned} f(t) &= \frac{a}{2}(t-1)(t-2) - bt(t-2) + \frac{c}{2}t(t-1) \\ &= \left(\frac{a}{2} - b + \frac{c}{2}\right)t^2 + \left(-\frac{3a}{2} + 2b - \frac{c}{2}\right)t + a. \end{aligned}$$

(e) What happens if we consider instead the linear map $S'\colon \mathcal{P}_1 \to \mathbb{R}^3$ defined by the same formula? (Here we are restricting the domain of S to the polynomials of degree at most 1.) Clearly, $\ker(S') = \{\mathbf{0}\}$, but now which vectors $(a, b, c) \in \mathbb{R}^3$ are in $\text{image}\,(S')$? We leave it to the reader to check that $(a, b, c) \in \text{image}\,(S')$ if and only if $a - 2b + c = 0$. ▲

Reviewing some terminology from the blue box on p. 88, we say that the linear map $T\colon V \to W$ is *onto* (or surjective) when $\text{image}\,(T) = W$. This means that for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$. When $T = \mu_A$ for an $m \times n$ matrix A, this corresponds to saying that $\mathbf{C}(A) = \mathbb{R}^m$, which we know occurs precisely when A has rank m. On the other hand, $\ker(T) = \{\mathbf{0}\}$ happens precisely when solutions of $T(\mathbf{v}) = \mathbf{w}$ are unique, for if $T(\mathbf{v}_1) = \mathbf{w}$ and $T(\mathbf{v}_2) = \mathbf{w}$, then $T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$, so $\mathbf{v}_1 = \mathbf{v}_2$ if and only if $\mathbf{0}$ is the only vector in $\ker(T)$. In this case, we say that T is *one-to-one* (or injective).

When a linear transformation $T\colon V \to W$ is both one-to-one and onto, it gives a one-to-one correspondence between the elements of V and the elements of W. Moreover, because T is a linear map, this correspondence respects the linear structure of the two vector spaces; that is, if $\mathbf{v}_1$ and $\mathbf{v}_2$ correspond to $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively, then $a\mathbf{v}_1 + b\mathbf{v}_2$ corresponds to $a\mathbf{w}_1 + b\mathbf{w}_2$ for any scalars a and b. Thus, for all intents and purposes, T provides a complete dictionary translating the elements (and the algebraic structure) of V into those of W, and the two spaces are "essentially" the same. This leads us to the following definition.

Definition. A linear map $T\colon V \to W$ that is both one-to-one and onto is called an *isomorphism*.[8]

This definition leads us naturally to the following proposition.

Proposition 4.1. *A linear transformation $T\colon V \to W$ is an isomorphism if and only if there is a linear transformation $T^{-1}\colon W \to V$ satisfying $(T^{-1} \circ T)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ and $(T \circ T^{-1})(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$.*

Proof. Suppose we have such a linear transformation $T^{-1}\colon W \to V$. Then we will show that $\ker(T) = \{\mathbf{0}\}$ and $\text{image}\,(T) = W$. First, suppose that $T(\mathbf{v}) = \mathbf{0}$. Then, applying the function T^{-1}, we have $\mathbf{v} = T^{-1}\big(T(\mathbf{v})\big) = T(\mathbf{0}) = \mathbf{0}$, so $\ker(T) = \{\mathbf{0}\}$.[9] Now, to establish that $\text{image}\,(T) = W$, we choose any $\mathbf{w} \in W$ and note that if we set $\mathbf{v} = T^{-1}(\mathbf{w})$, then we have $T(\mathbf{v}) = T(T^{-1}(\mathbf{w})) = \mathbf{w}$.

We leave the second half of the proof to the reader in Exercise 13. □

EXAMPLE 4

Given a finite-dimensional vector space V and an ordered basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V, we can define a function $C_{\mathcal{B}}\colon V \to \mathbb{R}^n$ that assigns to each vector $\mathbf{v} \in V$ its vector of coordinates with respect to $\mathcal{B}$. That is,

$$C_{\mathcal{B}}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Since $\mathcal{B}$ is a basis for V, for each $\mathbf{v} \in V$ there exist *unique* scalars $c_1, c_2, \dots, c_n$ so that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$; this means that $C_{\mathcal{B}}$ is a well-defined function. We leave it to the reader to check that $C_{\mathcal{B}}$ is a linear transformation. It is also one-to-one and onto (why?), and therefore $C_{\mathcal{B}}$ defines an isomorphism from V to $\mathbb{R}^n$.

Indeed, it follows from Proposition 4.1 that $C_{\mathcal{B}}^{-1}\colon \mathbb{R}^n \to V$, which associates to each n-tuple of coefficients the corresponding linear combination of the basis vectors,

$$C_{\mathcal{B}}^{-1}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$

is also a linear transformation. ▲

We see from the previous example that, given a basis for a finite-dimensional abstract vector space V, we can, using the isomorphism $C_{\mathcal{B}}$, identify V with $\mathbb{R}^n$ for the appropriate positive integer n. We will next use this identification to associate matrices to linear transformations between abstract vector spaces.

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[8] This comes from the Greek root *isos*, “equal,” and *morphē*, ”form” or “shape.”

[9] Of course, we are taking it for granted that $T(\mathbf{0}) = \mathbf{0}$ for any linear map T. This follows from the fact that $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$.

We saw in the previous section how to define the matrix for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to a basis $\mathcal{B}$ for $\mathbb{R}^n$. Although we didn't say so explicitly, we worked with the formula

$$[T]_{\mathcal{B}} C_{\mathcal{B}}(\mathbf{x}) = C_{\mathcal{B}}(T(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

We will now use the same idea to associate matrices to linear transformations on finite-dimensional abstract vector spaces, given a choice of ordered bases for domain and range.

Definition. Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for V, and let $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be an ordered basis for W. Define numbers a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, by

$$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m, \qquad j = 1, \dots, n.$$

Then we define $A = [T]_{\mathcal{V},\mathcal{W}} = \left[a_{ij}\right]$ to be the *matrix for T with respect to $\mathcal{V}$ and $\mathcal{W}$.*

Remark.

(a) We will usually assume, as in Section 3, that whenever the vector spaces V and W are the same, we will take the bases to be the same, i.e., $\mathcal{W} = \mathcal{V}$.

(b) Since $a_{1j}, \dots, a_{mj}$ are the coordinates of $T(\mathbf{v}_j)$ with respect to the basis $\mathcal{W}$, i.e., $C_{\mathcal{W}}(T(\mathbf{v}_j)) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, we can use the schematic notation as before:

$$A = [T]_{\mathcal{V},\mathcal{W}} = \begin{bmatrix} | & | & & | \\ C_{\mathcal{W}}(T(\mathbf{v}_1)) & C_{\mathcal{W}}(T(\mathbf{v}_2)) & \cdots & C_{\mathcal{W}}(T(\mathbf{v}_n)) \\ | & | & & | \end{bmatrix}.$$

(c) The calculation in the proof of Proposition 3.1 shows now that if $\mathbf{v} = \sum_{i=1}^{n} x_i \mathbf{v}_i$ (i.e., $C_{\mathcal{V}}(\mathbf{v}) = \mathbf{x}$) and $\mathbf{w} = \sum_{j=1}^{m} y_j \mathbf{w}_j$ (i.e., $C_{\mathcal{W}}(\mathbf{w}) = \mathbf{y}$), then

$$T(\mathbf{v}) = \mathbf{w} \quad \text{if and only if} \quad A\mathbf{x} = \mathbf{y}.$$

That is,

$$C_{\mathcal{W}}(T(\mathbf{v})) = AC_{\mathcal{V}}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V.$$

This can be summarized by the diagram in Figure 4.1. If we start with $\mathbf{v} \in V$, we can either go down and then to the right, obtaining $AC_{\mathcal{V}}(\mathbf{v}) = A\mathbf{x}$, or else go to the right and then down, obtaining $C_{\mathcal{W}}(T(\mathbf{v})) = \mathbf{y}$. The matrix A is defined so that we get the same answer either way.

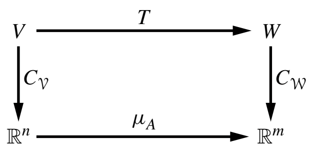

FIGURE 4.1

(d) What's more, suppose U, V, and W are vector spaces with bases $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$, respectively. Suppose also that A is the matrix for a linear transformation $T: V \to W$ with respect to $\mathcal{V}$ and $\mathcal{W}$, and suppose that B is the matrix for $S: U \to V$ with respect to $\mathcal{U}$ and $\mathcal{V}$. Then, because matrix multiplication corresponds to composition of linear transformations, AB is the matrix for $T \circ S$ with respect to $\mathcal{U}$ and $\mathcal{W}$.

EXAMPLE 5

Let's return now to $D: \mathcal{P}_3 \to \mathcal{P}_2$. Let's choose "standard" bases for these vector spaces: $\mathcal{V} = \{1, t, t^2, t^3\}$ and $\mathcal{W} = \{1, t, t^2\}$. Then

$$\begin{aligned} D(1) &= 0 &= 0\cdot 1 + 0\cdot t + 0\cdot t^2 \\ D(t) &= 1 &= 1\cdot 1 + 0\cdot t + 0\cdot t^2 \\ D(t^2) &= 2t &= 0\cdot 1 + 2\cdot t + 0\cdot t^2 \\ D(t^3) &= 3t^2 &= 0\cdot 1 + 0\cdot t + 3\cdot t^2. \end{aligned}$$

Now—and this is always the confusing part—we must be sure to arrange these coefficients as the *columns*, and *not* as the rows, of our matrix:

$$[D]_{\mathcal{V},\mathcal{W}} = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Let's make sure we understand what this means. Suppose $f(t) = 2 - t + 5t^2 + 4t^3 \in \mathcal{P}_3$ and we wish to calculate $D(f)$. The coordinate vector of f with respect to the basis $\mathcal{V}$ is

$$\begin{bmatrix} 2 \\ -1 \\ 5 \\ 4 \end{bmatrix},$$

and so

$$A\begin{bmatrix} 2 \\ -1 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \\ 12 \end{bmatrix},$$

which is the coordinate vector of $D(f) \in \mathcal{P}_2$ with respect to the basis $\mathcal{W}$. That is, $D(f) = -1 + 10t + 12t^2$. ▲

EXAMPLE 6

Define $T: \mathcal{P}_3 \to \mathcal{P}_4$ by $T(f)(t) = tf(t)$. Note that when we multiply a polynomial $f(t)$ of degree ≤ 3 by t, we obtain a polynomial of degree ≤ 4. We ask the reader to check that this is a linear transformation. The matrix for T with respect to the bases $\mathcal{V} = \{1, t, t^2, t^3\}$

and $\mathcal{W} = \{1, t, t^2, t^3, t^4\}$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

as we leave it to the reader to check. ▲

EXAMPLE 7

Define $T: \mathcal{P}_3 \to \mathbb{R}^2$ by $T(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$. Again we leave it to the conscientious reader to check that this is a linear transformation. With respect to the bases $\mathcal{V} = \{1, t, t^2, t^3\}$ for $\mathcal{P}_3$ and the standard basis $\mathcal{W} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$, the matrix for T is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$ ▲

EXAMPLE 8

Of course, if we use different bases for our vector spaces, then we will get different matrices representing our linear transformation. Returning to Example 5, let's instead use the basis $\mathcal{V}' = \{1, t-1, (t-1)^2, (t-1)^3\}$ for $\mathcal{P}_3$ and the same basis $\mathcal{W} = \{1, t, t^2\}$ for $\mathcal{P}_2$. Then

$$\begin{aligned} D(1) &= 0 &&= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 \\ D(t-1) &= 1 &&= 1 \cdot 1 + 0 \cdot t + 0 \cdot t^2 \\ D((t-1)^2) &= 2(t-1) &&= -2 \cdot 1 + 2 \cdot t + 0 \cdot t^2 \\ D((t-1)^3) &= 3(t-1)^2 &&= 3 \cdot 1 - 6 \cdot t + 3 \cdot t^2. \end{aligned}$$

Thus, the matrix for D with respect to the bases $\mathcal{V}'$ and $\mathcal{W}$ is

$$[D]_{\mathcal{V}',\mathcal{W}} = A' = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$ ▲

EXAMPLE 9

Fix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and define $T: \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $T(X) = MX$. Then properties of matrix multiplication tell us that T is a linear transformation. Using the basis $\mathcal{V}$ consisting of

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for $\mathcal{M}_{2\times 2}$, the matrix for T is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

as we leave it to the reader to check. ▲

The matrix we have been discussing for a linear transformation $T: V \to W$ depends on the choice of ordered bases $\mathcal{V}$ and $\mathcal{W}$ for V and W, respectively. If we choose alternative bases $\mathcal{V}'$ and $\mathcal{W}'$, our experience in Section 3, where we had $V = W = \mathbb{R}^n$ and $\mathcal{V} = \mathcal{W} = \mathcal{B}$, shows that we should expect a change-of-basis formula relating the two matrices. Let's now figure this out. Given ordered bases $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{V}' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$, define the *change-of-basis matrix from* $\mathcal{V}$ *to* $\mathcal{V}'$ as before: Let P be the $n \times n$ matrix whose j^{th} column vector consists of the coordinates of the vector $\mathbf{v}'_j$ with respect to the basis $\mathcal{V}$, i.e.,

$$\mathbf{v}'_j = p_{1j}\mathbf{v}_1 + p_{2j}\mathbf{v}_2 + \cdots + p_{nj}\mathbf{v}_n.$$

That is, we have the usual matrix P giving the change of basis in V:

$$P = \begin{bmatrix} | & | & & | \\ C_{\mathcal{V}}(\mathbf{v}'_1) & C_{\mathcal{V}}(\mathbf{v}'_2) & \cdots & C_{\mathcal{V}}(\mathbf{v}'_n) \\ | & | & & | \end{bmatrix}.$$

Then do the same thing with the bases for W: Let $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and $\mathcal{W}' = \{\mathbf{w}'_1, \ldots, \mathbf{w}'_m\}$ be two ordered bases for W, and let Q be the $m \times m$ matrix whose j^{th} column vector consists of the coordinates of the vector $\mathbf{w}'_j$ with respect to the basis $\mathcal{W}$, i.e.,

$$\mathbf{w}'_j = q_{1j}\mathbf{w}_1 + q_{2j}\mathbf{w}_2 + \cdots + q_{mj}\mathbf{w}_m.$$

So we now have the matrix Q giving the change of basis in W:

$$Q = \begin{bmatrix} | & | & & | \\ C_{\mathcal{W}}(\mathbf{w}'_1) & C_{\mathcal{W}}(\mathbf{w}'_2) & \cdots & C_{\mathcal{W}}(\mathbf{w}'_m) \\ | & | & & | \end{bmatrix}.$$

Then we have the following theorem.

Theorem 4.2 (Change-of-Basis Formula, Take 2). *Let V and W be finite-dimensional vector spaces, and let $T: V \to W$ be a linear transformation. Let $\mathcal{V}$ and $\mathcal{V}'$ be ordered bases for V, and let $\mathcal{W}$ and $\mathcal{W}'$ be ordered bases for W. Let P and Q be the change-of-basis matrices from $\mathcal{V}$ to $\mathcal{V}'$ and from $\mathcal{W}$ to $\mathcal{W}'$, respectively. If $A = [T]_{\mathcal{V},\mathcal{W}}$ and $A' = [T]_{\mathcal{V}',\mathcal{W}'}$, then we have*

$$A' = Q^{-1}AP.$$

This result is summarized in the diagram in Figure 4.2 (where we've omitted the μ's in the bottom rectangle for clarity).[10]

[10] This diagram seems quite forbidding at first blush, but it really does contain all the information in the theorem. You just need to follow the arrows around, composing functions, starting and ending at the appropriate places.

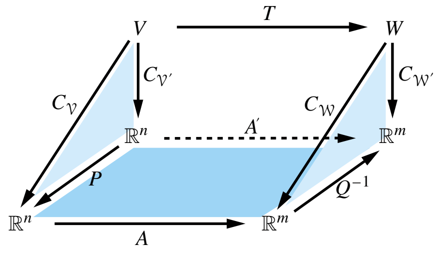

FIGURE 4.2

Proof. One can give a proof exactly like that of Proposition 3.2, and we leave this to the reader in Exercise 10. Here we give an argument that, we hope, explains the diagram.

Given a vector $\mathbf{v} \in V$, let $\mathbf{x} = C_{\mathcal{V}}(\mathbf{v})$ and $\mathbf{x}' = C_{\mathcal{V}'}(\mathbf{v})$. The important relation here is

$$\mathbf{x} = P\mathbf{x}'.$$

We derive this as follows. Using the equations $\mathbf{v} = \sum_{i=1}^{n} x_i \mathbf{v}_i$ and

$$\mathbf{v} = \sum_{j=1}^{n} x_j' \mathbf{v}_j' = \sum_{j=1}^{n} x_j' \Big(\sum_{i=1}^{n} p_{ij} \mathbf{v}_i \Big) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} p_{ij} x_j' \Big) \mathbf{v}_i,$$

we deduce from Corollary 3.3 of Chapter 3 that

$$x_i = \sum_{j=1}^{n} p_{ij} x_j'.$$

Likewise, if $T(\mathbf{v}) = \mathbf{w}$, let $\mathbf{y} = C_{\mathcal{W}}(\mathbf{w})$ and $\mathbf{y}' = C_{\mathcal{W}'}(\mathbf{w})$. As above, we will have $\mathbf{y} = Q\mathbf{y}'$. Now compare the equations $\mathbf{y}' = A'\mathbf{x}'$ and $\mathbf{y} = A\mathbf{x}$ using $\mathbf{x} = P\mathbf{x}'$ and $\mathbf{y} = Q\mathbf{y}'$: We have, on the one hand, $\mathbf{y} = Q\mathbf{y}' = Q(A'\mathbf{x}') = (QA')\mathbf{x}'$ and, on the other hand, $\mathbf{y} = A\mathbf{x} = A(P\mathbf{x}') = (AP)\mathbf{x}'$. Since $\mathbf{x}'$ is arbitrary, we conclude that $AP = QA'$, and so $A' = Q^{-1}AP$, as we wished to establish. □

EXAMPLE 10

We revisit Example 7, using instead the basis $\mathcal{V}' = \{1, t-1, t^2-t, t^3-t^2\}$ for $\mathcal{P}_3$ (why is this a basis?). We see directly that the matrix for T with respect to the bases $\mathcal{V}'$ and $\mathcal{W}$ is

$$A' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Does this agree with what we get applying Theorem 4.2? The change-of-basis matrix is

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $Q = I_{2\times 2}$ since we are not changing basis in $W = \mathbb{R}^2$. Therefore, we obtain

$$A' = Q^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

as we hoped. ▲

EXAMPLE 11

Returning to Example 9, with $V = W = \mathcal{M}_{2\times 2}$, we let $\mathcal{V} = \mathcal{W} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ as before and take the new basis

$$\mathbf{v}_1' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v}_3' = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4' = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Since $\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2' = \mathbf{v}_2 + \mathbf{v}_4$, $\mathbf{v}_3' = \mathbf{v}_1 - \mathbf{v}_3$, and $\mathbf{v}_4' = \mathbf{v}_2 - \mathbf{v}_4$, the change-of-basis matrix is

$$P = Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

and the reader can check that (again applying Exercise 4.2.11)

$$P^{-1} = Q^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Using the matrix A we obtained earlier, we now find that

$$A' = Q^{-1}AP = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We see that we can interpret T as a reflection of $\mathcal{M}_{2\times 2}$ across the two-dimensional plane spanned by $\mathbf{v}_1'$ and $\mathbf{v}_2'$. Notice that multiplying by M switches the rows of a 2×2 matrix, so, indeed, the matrices $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are left fixed, and the matrices $\mathbf{v}_3'$ and $\mathbf{v}_4'$ are multiplied by -1. ▲

EXAMPLE 12

Finally, consider the linear transformation $T\colon \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(f)(t) = f''(t) + 4f'(t) - 5f(t)$. We ask the reader to check, in Exercise 3, that T is in fact a linear map and that its matrix with respect to the "standard" basis $\mathcal{V} = \{1, t, t^2, t^3\}$ for $\mathcal{P}_3$ is

$$A = \begin{bmatrix} -5 & 4 & 2 & 0 \\ 0 & -5 & 8 & 6 \\ 0 & 0 & -5 & 12 \\ 0 & 0 & 0 & -5 \end{bmatrix}.$$

Because this matrix is already in echelon form, we see that $\mathbf{N}(A) = \{\mathbf{0}\}$ and $\mathbf{C}(A) = \mathbb{R}^4$. Thus, we infer from Exercise 12 that $\ker(T) = \{\mathbf{0}\}$ and $\text{image}\,(T) = \mathcal{P}_3$. ▲

Exercises 4.4

1. In each case, a linear transformation $T\colon \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ is defined. Give the matrix for T with respect to the "standard basis" $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $\mathcal{M}_{2\times 2}$. In each case, determine $\ker(T)$ and $\text{image}\,(T)$.

*a. $T(X) = X^{\mathsf{T}}$

*b. $T(X) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X$

c. $T(X) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X$

d. $T(X) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X - X \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. Let $V \subset \mathcal{C}^\infty(\mathbb{R})$ be the given subspace. Let $D\colon V \to V$ be the differentiation operator $D(f) = f'$. Give the matrix for D with respect to the given basis.

*a. $V = \text{Span}\,(1, e^x, e^{2x}, \dots, e^{nx})$

b. $V = \text{Span}\,(e^x, xe^x, x^2e^x, \dots, x^n e^x)$

***3.** Verify the details of Example 12.

4. Use the change-of-basis formula to find the matrix for the linear transformation $D\colon \mathcal{P}_3 \to \mathcal{P}_2$ (see Example 5) with respect to the indicated bases. Here $\mathcal{V}$ and $\mathcal{W}$ indicate the "standard bases," as in Example 5.

*a. $\mathcal{V}' = \{1, t-1, (t-1)^2, (t-1)^3\}$, $\mathcal{W}' = \mathcal{W}$

b. $\mathcal{V}' = \mathcal{V}$, $\mathcal{W}' = \{1, t-1, (t-1)^2\}$

c. $\mathcal{V}' = \{1, t-1, (t-1)^2, (t-1)^3\}$, $\mathcal{W}' = \{1, t-1, (t-1)^2\}$

5. Define $T\colon \mathcal{P}_3 \to \mathcal{P}_3$ by

$$T(f)(t) = 2f(t) + (1-t)f'(t).$$

a. Show that T is a linear transformation.

b. Give the matrix representing T with respect to the "standard basis" $\{1, t, t^2, t^3\}$.

c. Determine $\ker(T)$ and $\text{image}\,(T)$. Give your reasoning.

d. Let $g(t) = 1 + 2t$. Use your answer to part *b* to find a solution of the differential equation $T(f) = g$.

e. What are all the solutions of $T(f) = g$?

6. Consider the differentiation operator $D\colon \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ (or $\mathcal{P}_k \to \mathcal{P}_{k-1}$, if you prefer).

a. Show that $\ker(D) = \{\text{constant functions}\}$.

b. Give the interpretation of Theorem 5.3 of Chapter 1 familiar to all students of calculus.

7. Define $M\colon \mathcal{P} \to \mathcal{P}$ by $M(f)(t) = tf(t)$, and let $D\colon \mathcal{P} \to \mathcal{P}$ be the differentiation operator, as usual.

a. Calculate $D\circ M - M\circ D$.

b. Check your result of part *a* with matrices if you consider the transformation mapping the finite-dimensional subspace $\mathcal{P}_3$ to $\mathcal{P}_3$. (Remark: You will need to use the matrices for both $D\colon \mathcal{P}_3 \to \mathcal{P}_2$ and $D\colon \mathcal{P}_4 \to \mathcal{P}_3$, as well as the matrices for both $M\colon \mathcal{P}_2 \to \mathcal{P}_3$ and $M\colon \mathcal{P}_3 \to \mathcal{P}_4$.)

c. Show that there can be no linear transformations $S\colon V \to V$ and $T\colon V \to V$ on a *finite-dimensional* vector space V with the property that $S\circ T - T\circ S = I$. (*Hint:* See Exercise 3.6.9.) Why does this not contradict the result of part *b*?

8. Let V and W be vector spaces, and let $T\colon V \to W$ be a linear transformation.

*a. Show that T maps the line through $\mathbf{u}$ and $\mathbf{v}$ to the line through $T(\mathbf{u})$ and $T(\mathbf{v})$. What does this mean if $T(\mathbf{u}) = T(\mathbf{v})$?

b. Show that T maps parallel lines to parallel lines.

9. a. Consider the identity transformation $\mathrm{Id}\colon \mathbb{R}^n \to \mathbb{R}^n$. Using the basis $\mathcal{V}$ in the domain and the basis $\mathcal{V}'$ in the range, show that the matrix $[\mathrm{Id}]_{\mathcal{V},\mathcal{V}'}$ is the inverse of the change-of-basis matrix P.

b. Use this observation to give another derivation of the change-of-basis formula.

10. Give a proof of Theorem 4.2 modeled on the proof of Proposition 3.2.

11. Let V and W be vector spaces (not necessarily finite-dimensional), and let $T\colon V \to W$ be a linear transformation. Check that $\ker(T) \subset V$ and $\text{image}\,(T) \subset W$ are subspaces.

12. Suppose $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an ordered basis for V, $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is an ordered basis for W, and A is the matrix for the linear transformation $T\colon V \to W$ with respect to these bases.

a. Check that $\mathbf{x} \in \mathbf{N}(A) \iff x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \in \ker(T)$.

b. Check that $\mathbf{y} \in \mathbf{C}(A) \iff y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m \in \text{image}\,(T)$.

13. Prove that if a linear transformation $T\colon V \to W$ is an isomorphism, then there is a linear transformation $T^{-1}\colon W \to V$ satisfying $(T^{-1}\circ T)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$ and $(T\circ T^{-1})(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$.

14. Decide whether each of the following functions T is a linear transformation. If not, explain why. If so, give $\ker(T)$ and $\text{image}\,(T)$.

*a. $T\colon \mathbb{R}^n \to \mathbb{R}$, $\quad T(\mathbf{x}) = \|\mathbf{x}\|$

*b. $T\colon \mathcal{P}_3 \to \mathbb{R}$, $\quad T(f) = \int_0^1 f(t)\,dt$

c. $T\colon \mathcal{M}_{m\times n} \to \mathcal{M}_{n\times m}$, $\quad T(X) = X^{\mathsf{T}}$

*d. $T\colon \mathcal{P} \to \mathcal{P}$, $\quad T(f)(t) = \int_0^t f(s)\,ds$

e. $T\colon \mathcal{C}^0(\mathbb{R}) \to \mathcal{C}^1(\mathbb{R})$, $\quad T(f)(t) = \int_0^t f(s)\,ds$

15. Let $\mathcal{I} = [0, 1]$, and let $M\colon \mathcal{C}^0(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I})$ be given by $M(f)(t) = tf(t)$. Determine $\ker(T)$ and $\text{image}\,(T)$.

16. Let V be a finite-dimensional vector space, let W be a vector space, and let $T: V \to W$ be a linear transformation. Give a matrix-free proof that $\dim(\ker T) + \dim(\text{image } T) = \dim V$, as follows.

a. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $\ker T$, and (following Exercise 3.4.17) extend to obtain a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for V. Show that $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ gives a basis for image (T).

b. Conclude the desired result. Explain why this is a restatement of Corollary 4.7 of Chapter 3 when W is finite-dimensional.

17. Suppose $T: V \to W$ is an isomorphism and $\dim V = n$. Prove that $\dim W = n$.

18. a. Suppose $T: V \to W$ is a linear transformation. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$ is linearly dependent. Prove that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\} \subset W$ is linearly dependent.

b. Suppose $T: V \to V$ is a linear transformation and V is finite-dimensional. Suppose image $(T) = V$. Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset V$ is linearly independent, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is linearly independent. (*Hint:* Use Exercise 12 or Exercise 16.)

19. Let V and W be subspaces of $\mathbb{R}^n$ with $V \cap W = \{\mathbf{0}\}$. Let $S = \text{proj}_V$ and $T = \text{proj}_W$. Show that $S \circ T = T \circ S$ if and only if V and W are orthogonal subspaces. (They need not be orthogonal complements, however.)

20. Suppose V is a vector space and $T: V \to V$ is a linear transformation. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ are nonzero vectors satisfying

$$
\begin{aligned}
T(\mathbf{v}_1) &= \mathbf{v}_1 \\
T(\mathbf{v}_2) &= 2\mathbf{v}_2 \\
T(\mathbf{v}_3) &= -\mathbf{v}_3.
\end{aligned}
$$

Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

21. Let V be a vector space.

a. Let V^* denote the set of all linear transformations from V to $\mathbb{R}$. Show that V^* is a vector space.

b. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V. For $i = 1, \dots, n$, define $\mathbf{f}_i \in V^*$ by

$$\mathbf{f}_i(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_i.$$

Prove that $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ gives a basis for V^*.

c. Deduce that whenever V is finite-dimensional, $\dim V^* = \dim V$.

22. Let $t_1, \dots, t_{k+1}$ be distinct real numbers. Define a linear transformation $T: \mathcal{P}_k \to \mathbb{R}^{k+1}$ by

$$T(f) = \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_{k+1}) \end{bmatrix}.$$

a. Prove that $\ker(T) = \{\mathbf{0}\}$.

b. Show that the matrix for T with respect to the "standard bases" for $\mathcal{P}_k$ and $\mathbb{R}^{k+1}$ is the matrix A on p. 185.

c. Deduce that the matrix A on p. 185 is nonsingular. (See Exercise 12.)

d. Explain the origins of the inner product on $\mathcal{P}_k$ defined in Example 10(b) in Section 6 of Chapter 3.

23. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ has the following properties:

(i) $T(\mathbf{0}) = \mathbf{0}$;

(ii) T preserves distance (i.e., $\|T(\mathbf{x}) - T(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$).

a. Prove that $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

b. If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis, let $T(\mathbf{e}_i) = \mathbf{v}_i$. Prove that $T\left(\sum_{i=1}^{n} x_i \mathbf{e}_i\right) = \sum_{i=1}^{n} x_i \mathbf{v}_i$.

c. Deduce from part *b* that T is a linear transformation.

d. Prove that the standard matrix for T is orthogonal.

24. (See the discussion on p. 167 and Exercise 3.4.25.) Let A be an $n \times n$ matrix. Prove that the functions $\mu_A : \mathbf{R}(A) \to \mathbf{C}(A)$ and $\mu_{A^\mathsf{T}} : \mathbf{C}(A) \to \mathbf{R}(A)$ are inverse functions if and only if $A = QP$, where P is a projection matrix and Q is orthogonal.

HISTORICAL NOTES

Carl Friedrich Gauss (1777–1855) invented the method of least squares while he was studying the orbits of asteroids. In 1801 he successfully predicted the orbit of Ceres, an asteroid discovered by the Italian astronomer G. Piazzi on the first day of that year. The work was so impressive that the German astronomer Wilhelm Olbers asked him to apply his methods to study the second known asteroid, Pallas, which Olbers had discovered in 1802. In a paper of 1809, Gauss summarized his work on Pallas. His calculations had led him to an inconsistent linear system in six variables, which therefore required a least-squares approach. In the same paper he also used the techniques of what we now call Gaussian elimination. Adrien-Marie Legendre (1752–1833) actually published the least squares method first, in 1806, in a book describing methods for determining the orbits of comets. Gauss claimed priority in his book, and it is now generally accepted that he was the first to design and use the method.

The uses of orthogonality and orthogonal projections go far beyond the statistical analysis of data. Jean Baptiste Joseph Fourier (1768–1830) was a great French mathematician who focused much of his attention on understanding and modeling physical phenomena such as heat transfer and vibration. His work led to what is now called Fourier series and the approach to problem solving called Fourier analysis. Fourier was a mathematical prodigy and studied at the École Royale Militaire in his hometown of Auxerre. A short time after leaving school, he applied to enter the artillery or the engineers, but was denied. He then decided, at the age of nineteen, to join an abbey to study for the priesthood. He never took his religious vows; instead, he was offered a professorship at the École Militaire, and mathematics became his life's work. He was imprisoned during the Reign of Terror in 1794 and may have come close to being guillotined. He accepted a job at the École Normale in Paris, starting in 1795, and soon thereafter was offered the chair of analysis at the École Polytechnique. He accompanied Napoleon as scientific adviser to Egypt in 1798 and was rewarded by being appointed governor of lower Egypt. When he returned to Paris in 1801, he was appointed prefect of the Department of Isère.

During his time in Grenoble, Fourier developed his theory of heat, completing his memoir *On the Propagation of Heat in Solid Bodies* (1807). Fourier's ideas were met with a cold reception when he proposed them in the early nineteenth century, but now his techniques are indispensable in fields such as electrical engineering. The fundamental idea behind Fourier analysis is that many functions can be approximated by linear combinations of the functions 1, $\cos nx$, and $\sin nx$, as n ranges over all positive integers. These functions are orthogonal in the vector space of continuous functions on the interval $[0, 2\pi]$ endowed with the inner product described in Example 10(c) in Section 6 of Chapter 3. Thus, the basic idea of Fourier analysis is that continuous functions should be well approximated by their projections onto (finite-dimensional) spans of these orthogonal functions. Other classes of orthogonal functions have been studied over the years and applied to the theory of differential equations by mathematicians such as Legendre, Charles Hermite (1822–1901), Pafnuty Lvovich Chebyshev (1821–1894), and Edmond Nicolas Laguerre (1834–1886). Many of these ideas have also played a significant role in the modern theory of interpolation and numerical integration.

CHAPTER 5

DETERMINANTS

To each square matrix we associate a number, called its determinant. For us, that number will provide a convenient computational criterion for determining whether a matrix is singular. We will use this criterion in Chapter 6. The determinant also has a geometric interpretation in terms of area and volume. Although this interpretation is not necessary for our later work, we find it such a beautiful example of the interplay between algebra and geometry that we could not resist telling at least part of this story. Those who study multivariable calculus will recognize the determinant when it appears in the change-of-variables formula for integrals.

1 Properties of Determinants

In Section 3 of Chapter 2 we saw that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if the quantity $ad - bc$ is nonzero. Here we give this quantity a name, the *determinant* of A, denoted $\det A$; i.e., $\det A = ad - bc$. We will explore the geometric interpretation of the determinant in Section 3, but for now we want to figure out how this idea should generalize to $n \times n$ matrices. To do this, we will study the effect of row operations on the determinant. If we switch the two rows, the determinant changes sign. If we multiply one of the rows by a scalar, the determinant multiplies by that same factor. Slightly less obvious is the observation that if we do a row operation of type (iii)—adding a multiple of one row to the other—then the determinant does not change: For any scalar k, consider that

$$\det \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix} = a(d + kb) - b(c + ka) = ad - bc = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Now let's reverse this reasoning. Starting with these three properties and the requirement that the determinant of the identity matrix be 1, can we derive the formula $\det A = ad - bc$? Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and assume that $a \neq 0$. First we add $-c/a$ times the first row to the second:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}.$$

So far, the determinant has not changed. If $ad - bc = 0$, then both entries of the second row are 0, and so the determinant is 0 (why?). Provided that $ad - bc \neq 0$, we can add a suitable multiple of the second row to the first row to obtain

$$\begin{bmatrix} a & 0 \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.$$

But this matrix is obtained from the identity matrix by multiplying the first row by a and the second row by $\frac{ad-bc}{a}$, so we start with the determinant of the identity matrix, 1, and multiply by $a \cdot \frac{ad-bc}{a} = ad - bc$, and this is the determinant of A. The persnickety reader may wonder what to do if $a = 0$. If $c \neq 0$, we switch rows and get a determinant of $-bc$; and last, if $a = c = 0$, we can arrange, by a suitable row operation of type (iii) if necessary, for a zero row, so once again the determinant is 0.

It turns out that this story generalizes to $n \times n$ matrices. We will prove in the next section that there is a unique function, det, that assigns to each $n \times n$ matrix a real number, called its determinant, that is characterized by the effect of elementary row operations and by its value on the identity matrix. We state these properties in the following proposition.

Proposition 1.1. *Let A be an $n \times n$ matrix.*

1. *Let A' be obtained from A by exchanging two rows. Then* $\det A' = -\det A$.
2. *Let A' be obtained from A by multiplying some row by the scalar c. Then* $\det A' = c \det A$.
3. *Let A' be obtained from A by adding a multiple of one row to another. Then* $\det A' = \det A$.
4. *Last,* $\det I_n = 1$.

EXAMPLE 1

We now use row operations to calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

First we exchange rows 1 and 3, and then we proceed to echelon form:

$$\det A = \det \begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 4 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= -12 \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = -12 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -12,$$

where we've used the pivots to clear out the upper entries in the columns without changing the determinant. ▲

Generalizing the determinant criterion for two vectors in $\mathbb{R}^2$ to be linearly (in)dependent, we deduce the following characterization of nonsingular matrices that will be critical in Chapter 6.

Theorem 1.2. *Let A be a square matrix. Then A is nonsingular if and only if* $\det A \neq 0$.

Proof. Suppose A is nonsingular. Then its reduced echelon form is the identity matrix. Turning this upside down, we can start with the identity matrix and perform a sequence of row operations to obtain A. If we keep track of the effect on the determinant, we see that we've started with $\det I = 1$ and multiplied it by a nonzero number to obtain $\det A$. That is, $\det A \neq 0$. Conversely, suppose A is singular. Then its echelon form U has a row of zeroes, and therefore $\det U = 0$ (see Exercise 2). It follows, as in the previous case, that $\det A = 0$. □

We give next some properties of determinants that can be useful on both computational and theoretical grounds.

Proposition 1.3. *If A is an upper (lower) triangular $n \times n$ matrix, then* $\det A = a_{11}a_{22}\cdots a_{nn}$*; that is,* $\det A$ *is the product of the diagonal entries.*

Proof. If $a_{ii} = 0$ for some i, then A is singular (why?) and so $\det A = 0$, and the desired equality holds in this case. Now assume all the a_{ii} are nonzero. Let $\mathbf{A}_i$ be the i^{th} row vector of A, as usual, and write $\mathbf{A}_i = a_{ii}\mathbf{B}_i$, where the i^{th} entry of $\mathbf{B}_i$ is 1. Then, letting B be the matrix with rows $\mathbf{B}_i$ and using property **2** of Proposition 1.1 repeatedly, we have $\det A = a_{11}a_{22}\cdots a_{nn}\det B$. Now B is an upper (lower) triangular matrix with 1's on the diagonal, so, using property **3**, we can use the pivots to clear out the upper (lower) entries without changing the determinant; thus, $\det B = \det I = 1$. And so finally, $\det A = a_{11}a_{22}\cdots a_{nn}$, as promised. □

Remark. One must be careful to apply Proposition 1.3 *only* when the matrix is triangular. When there are nonzero entries on both sides of the diagonal, further work is required.

Of special interest is the "product rule" for determinants. Notice, first of all, that because row operations can be represented by multiplication by elementary matrices, Proposition 1.1 can be restated as follows:

Proposition 1.4. *Let E be an elementary matrix, and let A be an arbitrary square matrix. Then*

$$\det(EA) = \det E \det A.$$

Proof. Left to the reader in Exercise 3. □

Theorem 1.5. *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det A \det B.$$

Proof. Suppose A is singular, so that there is some nontrivial linear relation among its row vectors:

$$c_1\mathbf{A}_1 + \cdots + c_n\mathbf{A}_n = \mathbf{0}.$$

Then, multiplying by B on the right, we find that

$$c_1(\mathbf{A}_1 B) + \cdots + c_n(\mathbf{A}_n B) = \mathbf{0},$$

from which we conclude that there is (the same) nontrivial linear relation among the row vectors of AB, and so AB is singular as well. We infer from Theorem 1.2 that both $\det A = 0$ and $\det AB = 0$, and so the result holds in this case.

Now, if A is nonsingular, we know that we can write A as a product of elementary matrices, *viz.*, $A = E_m \cdots E_2 E_1$. We now apply Proposition 1.4 twice. First, we have

$$\det A = \det(E_m \cdots E_2 E_1) = \det E_m \cdots \det E_2 \det E_1;$$

but then we have

$$\begin{aligned}\det AB &= \det(E_m \cdots E_2 E_1 B) = \det E_m \cdots \det E_2 \det E_1 \det B \\ &= (\det E_m \cdots \det E_2 \det E_1) \det B = \det A \det B,\end{aligned}$$

as claimed. □

A consequence of this proposition is that $\det(AB) = \det(BA)$, even though matrix multiplication is not commutative. Another useful observation is the following:

Corollary 1.6. *If A is nonsingular, then* $\det(A^{-1}) = \dfrac{1}{\det A}$.

Proof. From the equation $AA^{-1} = I$ and Theorem 1.5 we deduce that $\det A \det(A^{-1}) = 1$, so $\det(A^{-1}) = 1/\det A$. □

A fundamental consequence of the product rule is the fact that similar matrices have the same determinant. (Recall that B is similar to A if $B = P^{-1}AP$ for some invertible matrix P.) For, using Theorem 1.5 and Corollary 1.6, we obtain

$$\det(P^{-1}AP) = \det(P^{-1})\det(AP) = \det(P^{-1})\det A \det P = \det A.$$

As a result, when V is a finite-dimensional vector space, it makes sense to define the determinant of a linear transformation $T : V \to V$. One writes down the matrix A for T with respect to *any* (ordered) basis and defines $\det T = \det A$. The Change-of-Basis Formula, Proposition 3.2 of Chapter 4, tells us that any two matrices representing T are similar and hence, by our calculation, have the same determinant. What's more, as we shall see shortly, $\det T$ has a nice geometric meaning: It gives the factor by which signed volume is distorted under the mapping by T.

The following result is somewhat surprising, as it tells us that whatever holds for rows must also hold for columns.

Proposition 1.7. *Let A be a square matrix. Then*

$$\det(A^{\mathsf{T}}) = \det A.$$

Proof. Suppose A is singular. Then so is A^{T} (see Exercise 3.4.12). Thus, $\det(A^{\mathsf{T}}) = 0 = \det A$, and so the result holds in this case. Suppose now that A is nonsingular. As in the preceding proof, we write $A = E_m \cdots E_2 E_1$. Now we have $A^{\mathsf{T}} = (E_m \cdots E_2 E_1)^{\mathsf{T}} = E_1^{\mathsf{T}} E_2^{\mathsf{T}} \cdots E_m^{\mathsf{T}}$, and so, using the product rule and the fact that $\det(E_i^{\mathsf{T}}) = \det E_i$ (see Exercise 4), we obtain

$$\det(A^{\mathsf{T}}) = \det(E_1^{\mathsf{T}})\det(E_2^{\mathsf{T}})\cdots\det(E_m^{\mathsf{T}}) = \det E_1 \det E_2 \cdots \det E_m = \det A. \qquad \square$$

An immediate and useful consequence of Proposition 1.7 is the fact that the determinant behaves the same under *column* operations as it does under row operations. We have

Corollary 1.8. *Let A be an $n \times n$ matrix.*

1. *Let A' be obtained from A by exchanging two* columns. *Then* $\det A' = -\det A$.
2. *Let A' be obtained from A by multiplying some* column *by the number c. Then* $\det A' = c \det A$.
3. *Let A' be obtained from A by adding a multiple of one* column *to another. Then* $\det A' = \det A$.

1.1 Linearity in Each Row

It is more common to start with a different version of Proposition 1.1. We replace property **3** with the following:

3′. Suppose the i^{th} row of the matrix A is written as a sum of two vectors, $\mathbf{A}_i = \mathbf{A}_i' + \mathbf{A}_i''$. Let A' denote the matrix with $\mathbf{A}_i'$ as its i^{th} row and all other rows the same as those of A, and likewise for A''. Then $\det A = \det A' + \det A''$.

Properties **2** and **3′** say that the determinant is a *linear* function of each of the row vectors of the matrix. Be careful! This is not the same as saying that $\det(A + B) = \det A + \det B$, which is false for most matrices A and B.

We will prove the following result at the end of Section 2.

Theorem 1.9. *For each $n \geq 1$, there is exactly one function* det *that associates to each $n \times n$ matrix a real number and has the properties* **1**, **2**, **3′**, *and* **4**.

The next two results establish the fact that properties **1**, **2**, and **3′** imply property **3**, so that all the results of this section will hold if we assume that the determinant satisfies property **3′** instead of property **3**. For the rest of this section, we assume that we know that $\det A$ satisfies properties **1**, **2**, and **3′**, and we use those properties to establish property **3**.

Lemma 1.10. *If two rows of a matrix A are equal, then* $\det A = 0$.

Proof. If $\mathbf{A}_i = \mathbf{A}_j$, then the matrix is unchanged when we switch rows i and j. On the other hand, by property **1**, the determinant changes sign when we switch these rows. That is, we have $\det A = -\det A$. This can happen only when $\det A = 0$. □

Now we can easily deduce property **3** from properties **1**, **2**, and **3′**:

Proposition 1.11. *Let A be an $n \times n$ matrix, and let B be the matrix obtained by adding a multiple of one row of A to another. Then* $\det B = \det A$.

Proof. Suppose B is obtained from A by replacing the i^{th} row by its sum with c times the j^{th} row; i.e., $\mathbf{B}_i = \mathbf{A}_i + c\mathbf{A}_j$, with $i \neq j$. By property **3′**, $\det B = \det A + \det A'$, where $\mathbf{A}_i' = c\mathbf{A}_j$ and all the other rows of A' are the corresponding rows of A. If we define the matrix A'' by setting $\mathbf{A}_i'' = \mathbf{A}_j$ and keeping all the other rows the same, then property **2** tells us that $\det A' = c \det A''$. But two rows of the matrix A'' are identical, so, by Lemma 1.10, $\det A'' = 0$. Therefore, $\det B = \det A$, as desired. □

Exercises 5.1

1. Calculate the following determinants.

a. $\det \begin{bmatrix} -1 & 3 & 5 \\ 6 & 4 & 2 \\ -2 & 5 & 1 \end{bmatrix}$

*b. $\det \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & -2 & 2 & 3 \\ 0 & 0 & 6 & 2 \end{bmatrix}$

c. $\det \begin{bmatrix} 1 & 4 & 1 & -3 \\ 2 & 10 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

*d. $\det \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

2. Suppose one row of the $n \times n$ matrix A consists only of 0 entries, i.e., $\mathbf{A}_i = \mathbf{0}$ for some i. Use the properties of determinants to show that $\det A = 0$.

3. Prove Proposition 1.4.

$^\sharp$**4.** Without using Proposition 1.7, show that for any elementary matrix E, we have $\det E^{\mathsf{T}} = \det E$. (*Hint:* Consider each of the three types of elementary matrices.)

5. Let A be an $n \times n$ matrix and let c be a scalar. Show that $\det(cA) = c^n \det A$.

6. Given that 1898, 3471, 7215, and 8164 are all divisible by 13, use properties of determinants to show that

$$\det\begin{bmatrix} 1 & 8 & 9 & 8 \\ 3 & 4 & 7 & 1 \\ 7 & 2 & 1 & 5 \\ 8 & 1 & 6 & 4 \end{bmatrix}$$

is divisible by 13. (*Hint:* Use Corollary 1.8. You may also use the result of Exercise 5.2.6—the determinant of a matrix with integer entries is an integer.)

$^\sharp$**7.** Let A be an $n \times n$ matrix. Show that

$$\det\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right] = \det A.$$

8. a. Show that

$$\det\begin{bmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{bmatrix} = (c-b)(d-b)(d-c).$$

b. Show that

$$\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

*c. In general, evaluate (with proof)

$$\det\begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^k \\ 1 & t_2 & t_2^2 & \dots & t_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{k+1} & t_{k+1}^2 & \dots & t_{k+1}^k \end{bmatrix}.$$

(See Exercises 3.6.12 and 4.4.22.)

9. Generalizing Exercise 7, we have:

♯a. Suppose $A \in \mathcal{M}_{k\times k}$, $B \in \mathcal{M}_{k\times \ell}$, and $D \in \mathcal{M}_{\ell\times\ell}$. Prove that

$$\det \left[\begin{array}{c|c} A & B \\ \hline O & D \end{array}\right] = \det A \det D.$$

b. Suppose now that A, B, and D are as in part *a* and $C \in \mathcal{M}_{\ell\times k}$. Prove that if A is invertible, then

$$\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \det A \det(D - CA^{-1}B).$$

c. If we assume, moreover, that $k = \ell$ and $AC = CA$, then deduce that

$$\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \det(AD - CB).$$

d. Give an example to show that the result of part *c* needn't hold when A and C do not commute.

*__10.__ Suppose A is an *orthogonal* $n \times n$ matrix. (Recall that this means that $A^{\mathsf{T}}A = I_n$.) What are the possible values of $\det A$?

11. Suppose A is a skew-symmetric $n \times n$ matrix. (Recall that this means that $A^{\mathsf{T}} = -A$.) Show that when n is odd, $\det A = 0$. Give an example to show that this needn't be true when n is even. (*Hint:* Use Exercise 5.)

12. Prove directly that properties **1**, **2**, and **3** imply property **3′** for the last row.

a. Do the case of 2×2 matrices first. Suppose first that $\{\mathbf{A}_1, \mathbf{A}_2'\}$ is linearly dependent; show that in this case, $\det A = \det A''$. Next, deduce the result when $\{\mathbf{A}_1, \mathbf{A}_2'\}$ is linearly independent by writing $\mathbf{A}_2''$ as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2'$.

b. Generalize this argument to $n \times n$ matrices. Consider first the case that $\{\mathbf{A}_1, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n'\}$ is linearly dependent, then the case that $\{\mathbf{A}_1, \dots, \mathbf{A}_{n-1}, \mathbf{A}_n'\}$ is linearly independent.

13. Using Proposition 1.4, prove the uniqueness statement in Theorem 1.9. That is, prove that the determinant function is uniquely determined by the properties **1**, **2**, **3′**, and **4**. (*Hint:* Mimic the proof of Theorem 1.5. It might be helpful to consider two functions $\det$ and $\widetilde{\det}$ that have these properties and to show that $\det(A) = \widetilde{\det}(A)$ for every square matrix A.)

2 Cofactors and Cramer's Rule

We began this chapter with a succinct formula for the determinant of a 2×2 matrix:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

and we showed that this function on $\mathcal{M}_{2\times 2}$ satisfies the properties listed in Proposition

1.1. Similarly, it is not too hard to show that it satisfies property **3′**; thus, this formula establishes the existence part of Theorem 1.9 in the case $n = 2$. It would be nice if there were a simple formula for the determinant of $n \times n$ matrices when $n > 2$. Reasoning as above, such a formula could help us prove that a function satisfying the properties in Theorem 1.9 actually *exists*.

Let's try to determine one in the case $n = 3$. Given the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we use the properties to calculate $\det A$. First we use linearity in the first row to break this up into the sum of three determinants:

$$\det A = a_{11} \det \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{12} \det \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} + a_{12} \det \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$(*) \qquad = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33},$$

where we've used the result of Exercise 5.1.7 and Corollary 1.8 at the last stage. Be sure to understand precisely how! The result can be depicted schematically as in Figure 2.1 below, but be warned that this handy mnemonic device works *only* for 3×3 determinants! (In

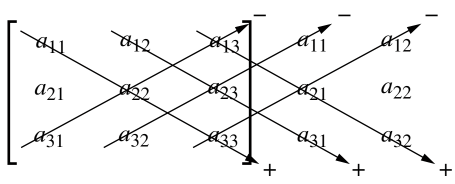

FIGURE 2.1

general, the determinant of an $n \times n$ matrix can be written as the sum of $n!$ terms, each ($\pm$) the product of n entries of the matrix, one from each row and column.)

Before proceeding to the $n \times n$ case, we make a few observations. First, although we "expanded" the above determinant along the first row, we could have expanded along any row. As we shall see in the following example, one must be a bit careful with signs. The reader might find it interesting to check that the final expression, $(*)$, for $\det A$ results from any of these three expansions. If we believe the uniqueness part of Theorem 1.9, this is no surprise. Second, since we know from the last section that $\det A = \det A^{\mathsf{T}}$, we can also "expand" the determinant along any column. Again, the reader may find it valuable to see that any such expansion results in the same final expression.

EXAMPLE 1

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix}.$$

Suppose we want to calculate det A by expanding along the second row. If we switch the first two rows, we get

$$\begin{aligned} \det A &= -\det \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \\ &= (-1)\left((1)\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - (-2)\det\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + (3)\det\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right) \\ &= (-1)\big((1)(-5) + (2)(2) + (3)(4)\big) = -11. \end{aligned}$$

Of course, because of the 0 entry in the third row, we'd have been smarter to expand along the third row. Now if we switch the first and third rows and then the second and third rows, the original rows will be in the order 3, 1, 2, and the determinant will be the same:

$$\begin{aligned} \det A &= \det \begin{bmatrix} 0 & 2 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \\ &= (0)\det\begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix} - (2)\det\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} + (1)\det\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \\ &= -2(3) + 1(-5) = -11. \end{aligned}$$

▲

The preceding calculations of a 3×3 determinant suggest a general recursive formula. Given an $n \times n$ matrix A with $n \geq 2$, denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column from A. Define the ij^{th} *cofactor* of the matrix to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Note that we include the coefficient of ± 1 according to the "checkerboard" pattern as indicated below:[1]

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then we have the following formula, which is called the *expansion in cofactors along the* i^{th} *row*.

[1]We can account for the power of -1 as follows, generalizing the procedure in Example 1: To move the i^{th} row to the top, without otherwise changing the order of the rows, requires switching pairs of rows $i-1$ times; this gives a sign of $(-1)^{i-1}$. We then alternate signs as we proceed from column to column, the j^{th} column contributing a sign of $(-1)^{j-1}$. Thus, in the expansion, det A_{ij} appears with a factor of $(-1)^{i-1}(-1)^{j-1} = (-1)^{i+j}$.

Proposition 2.1. *Let A be an $n \times n$ matrix. Then for any fixed i, we have*

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

Note that when we define the determinant of a 1×1 matrix by the obvious rule

$$\det [a] = a,$$

Proposition 2.1 yields the familiar formula for the determinant of a 2×2 matrix.

If we accept Theorem 1.9, the proof of Proposition 2.1 follows exactly along the lines of the computation we gave in the 3×3 case above: Just expand along the i^{th} row using linearity (property **3**′). We leave the details of this to the reader. Now, by using the fact that $\det A^{\mathsf{T}} = \det A$, we also have the *expansion in cofactors along the j^{th} column*:

Proposition 2.2. *Let A be an $n \times n$ matrix. Then for any fixed j, we have*

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

As we mentioned above, we can turn this whole argument around to use these formulas to prove Theorem 1.9. The proof we give here is somewhat sketchy and quite optional; the reader who is familiar with mathematical induction may wish to make this proof more complete by using that tool.

Proof of Theorem 1.9. First, we can deduce from the reasoning of Section 1 that there can be only one such function, because, by reducing the matrix to echelon form by row operations, we are able to compute the determinant. (See Exercise 5.1.13.) Now, to establish existence, we will show that the formula given in Proposition 2.2 satisfies properties **1**, **2**, **3**′, and **4**.

We begin with property **1**. When we form a new matrix A' by switching two adjacent rows (say, rows k and $k+1$) of A, then whenever $i \neq k$ and $i \neq k+1$, we have $a'_{ij} = a_{ij}$ and $C'_{ij} = -C_{ij}$; on the other hand, when $i = k$, we have $a'_{kj} = a_{k+1,j}$ and $C'_{kj} = -C_{k+1,j}$; when $i = k+1$, we have $a'_{k+1,j} = a_{kj}$ and $C'_{k+1,j} = -C_{kj}$, so

$$\sum_{i=1}^{n} a'_{ij} C'_{ij} = -\sum_{i=1}^{n} a_{ij} C_{ij},$$

as required. We can exchange an arbitrary pair of rows by exchanging an *odd number* of adjacent pairs in succession (see Exercise 9), so the general result follows.

The remaining properties are easier to check. If we multiply the k^{th} row by c, then for $i \neq k$, we have $a'_{ij} = a_{ij}$ and $C'_{ij} = cC_{ij}$, whereas for $i = k$, we have $C'_{kj} = C_{kj}$ and $a'_{kj} = ca_{kj}$. Thus,

$$\sum_{i=1}^{n} a'_{ij} C'_{ij} = c \sum_{i=1}^{n} a_{ij} C_{ij},$$

as required. Suppose now that we replace the k^{th} row by the sum of two row vectors, *viz.*, $\mathbf{A}'_k = \mathbf{A}_k + \mathbf{A}''_k$. Then for $i \neq k$, we have $C'_{ij} = C_{ij} + C''_{ij}$ and $a'_{ij} = a_{ij} = a''_{ij}$. When

$i = k$, we likewise have $C'_{kj} = C_{kj} = C''_{kj}$, but $a'_{kj} = a_{kj} + a''_{kj}$. So

$$\sum_{i=1}^{n} a'_{ij}C'_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij} + \sum_{i=1}^{n} a''_{ij}C''_{ij},$$

as required. Verifying the fourth property is straightforward and is left to the reader. □

Remark. It is worth remarking that expansion in cofactors is an important theoretical tool, but a computational nightmare. Even using calculators and computers, to compute an $n \times n$ determinant by expanding in cofactors requires (approximately) $n!$ multiplications (and additions). On the other hand, to compute an $n \times n$ determinant by row reducing the matrix to upper triangular form requires slightly fewer than $\frac{1}{3}n^3$ multiplications (and additions). Now, $n!$ grows faster[2] than $(n/e)^n$, which gets large *much* faster than does n^3. Indeed, consider the following table displaying the number of operations required:

n	cofactors	row operations
2	2	2
3	6	8
4	24	20
5	120	40
6	720	70
7	5,040	112
8	40,320	168
9	362,880	240
10	3,628,800	330

Thus, we see that once $n > 4$, it is sheer folly to calculate a determinant by the cofactor method (unless almost all the entries of the matrix happen to be 0).

Having said that, we can see that the cofactor method is particularly effective when a variable is involved.

EXAMPLE 2

The main thrust of Chapter 6 will be to find, given a square matrix A, values of t for which the matrix $A - tI$ is singular. By Theorem 1.2, we want to find out when the determinant is 0. For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad \text{then} \quad A - tI = \begin{bmatrix} 3-t & 2 & -1 \\ 0 & 1-t & 2 \\ 1 & 1 & -1-t \end{bmatrix},$$

[2]The precise estimate comes from Stirling's formula, which states that the ratio of $n!$ to $\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ approaches 1 as $n \to \infty$. See Spivak, *Calculus*, 4th ed., p. 578.

and we can compute $\det(A - tI)$ by expanding in cofactors along the first column. Note that this saves us work because one of the terms will drop out.

$$\begin{aligned}\det(A - tI) &= \det\begin{bmatrix} 3-t & 2 & -1 \\ 0 & 1-t & 2 \\ 1 & 1 & -1-t \end{bmatrix} \\ &= (3-t)\det\begin{bmatrix} 1-t & 2 \\ 1 & -1-t \end{bmatrix} - (0)\det\begin{bmatrix} 2 & -1 \\ 1 & -1-t \end{bmatrix} \\ &\qquad + (1)\det\begin{bmatrix} 2 & -1 \\ 1-t & 2 \end{bmatrix} \\ &= (3-t)\big(-(1-t)(1+t) - 2\big) + \big(4 + (1-t)\big) \\ &= (3-t)(-3+t^2) + (5-t) \\ &= -t^3 + 3t^2 + 2t - 4. \end{aligned}$$

In Chapter 6 we will learn some tricks to find the roots of such polynomials. ▲

We conclude this section with a few classic formulas. The first is particularly useful for solving 2×2 systems of equations and may be useful even for larger n if you are interested only in a certain component x_i of the solution vector.

Proposition 2.3 Cramer's Rule. *Let A be a nonsingular $n \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^n$. Then the i^{th} coordinate of the vector $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$ is*

$$x_i = \frac{\det B_i}{\det A},$$

where B_i is the matrix obtained by replacing the i^{th} column of A by the vector $\mathbf{b}$.

Proof. This is amazingly simple. We calculate the determinant of the matrix obtained by replacing the i^{th} column of A by $\mathbf{b} = A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$:

$$\begin{aligned}\det B_i &= \det\begin{bmatrix} | & | & & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n & \cdots & \mathbf{a}_n \\ | & | & & | & & | \end{bmatrix} \\ &= \det\begin{bmatrix} | & | & & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & x_i\mathbf{a}_i & \cdots & \mathbf{a}_n \\ | & | & & | & & | \end{bmatrix} = x_i \det A, \end{aligned}$$

since the multiples of columns other than the i^{th} do not contribute to the determinant. □

EXAMPLE 3

We wish to solve

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

We have

$$B_1 = \begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix},$$

so $\det B_1 = 24$, $\det B_2 = -14$, and $\det A = 2$. Therefore, $x_1 = 12$ and $x_2 = -7$. ▲

We now deduce from Cramer's Rule an "explicit" formula for the inverse of a nonsingular matrix. Students seem always to want an alternative to Gaussian elimination, but what follows is practical only for the 2×2 case (where it gives us our familiar formula from Example 4 on p. 105) and—barely—for the 3×3 case. Having an explicit formula, however, can be useful for theoretical purposes.

Proposition 2.4. *Let A be a nonsingular matrix, and let $C = \left[C_{ij}\right]$ be the matrix of its cofactors. Then*

$$A^{-1} = \frac{1}{\det A} C^{\mathsf{T}}.$$

Proof. We recall from the discussion on p. 104 that the j^{th} column vector of A^{-1} is the solution of $A\mathbf{x} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} standard basis vector for $\mathbb{R}^n$. Now, Cramer's Rule tells us that the i^{th} coordinate of the j^{th} column of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \det \mathcal{A}_{ji},$$

where $\mathcal{A}_{ji}$ is the matrix obtained by replacing the i^{th} column of A by $\mathbf{e}_j$. Now, we calculate $\det \mathcal{A}_{ji}$ by expanding in cofactors along the i^{th} column of the matrix $\mathcal{A}_{ji}$. Since the only nonzero entry of that column is the j^{th}, and since all its remaining columns are those of the original matrix A, we find that

$$\det \mathcal{A}_{ji} = (-1)^{i+j} \det A_{ji} = C_{ji},$$

and this proves the result. □

EXAMPLE 4

Let's apply this result to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

without any row operations (compare with the answer obtained by Gaussian elimination on p. 105). First of all,

$$\det A = (1)\det\begin{bmatrix} -1 & 0 \\ -2 & 2 \end{bmatrix} - (-1)\det\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} + (1)\det\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = -1.$$

Next, we calculate the cofactor matrix. We leave it to the reader to check the details of the arithmetic. (Be careful not to forget the checkerboard pattern of $+$'s and $-$'s for the coefficients of the 2×2 determinants.)

$$C = \begin{bmatrix} -2 & -4 & -3 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Thus, applying Proposition 2.4, we have

$$A^{-1} = \frac{1}{\det A} C^{\mathsf{T}} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ 3 & -1 & -1 \end{bmatrix}.$$

▲

In fairness, for 3 × 3 matrices, this formula isn't bad when det A would cause troublesome arithmetic doing Gaussian elimination.

EXAMPLE 5

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix};$$

then

$$\det A = (1)\det\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - (2)\det\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} + (1)\det\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} = 15,$$

and so we suspect the fractions won't be fun if we do Gaussian elimination. Undaunted, we calculate the cofactor matrix:

$$C = \begin{bmatrix} 3 & 7 & -2 \\ -6 & 1 & 4 \\ 3 & -3 & 3 \end{bmatrix},$$

and so

$$A^{-1} = \frac{1}{\det A} C^{\mathsf{T}} = \frac{1}{15}\begin{bmatrix} 3 & -6 & 3 \\ 7 & 1 & -3 \\ -2 & 4 & 3 \end{bmatrix}.$$

▲

Exercises 5.2

1. Calculate the following determinants using cofactors.

a. $\det\begin{bmatrix} -1 & 3 & 5 \\ 6 & 4 & 2 \\ -2 & 5 & 1 \end{bmatrix}$

*b. $\det\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & -2 & 2 & 3 \\ 0 & 0 & 6 & 2 \end{bmatrix}$

c. $\det\begin{bmatrix} 1 & 4 & 1 & -3 \\ 2 & 10 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

*d. $\det\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

*__2.__ Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$.

a. If $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, use Cramer's Rule to find x_2.

b. Find A^{-1} using cofactors.

*__3.__ Using cofactors, find the determinant and the inverse of the matrix

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}.$$

4. Check that Proposition 2.4 gives the customary answer for the inverse of a nonsingular 2×2 matrix.

5. For each of the following matrices A, calculate $\det(A - tI)$.

*a. $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

*e. $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

*f. $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

h. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

*j. $\begin{bmatrix} 1 & -6 & 4 \\ -2 & -4 & 5 \\ -2 & -6 & 7 \end{bmatrix}$

6. Show that if the entries of a matrix A are integers, then $\det A$ is an integer. (*Hint:* Use induction.)

♯**7.** a. Suppose A is an $n \times n$ matrix with integer entries and $\det A = \pm 1$. Show that A^{-1} has all integer entries.

b. Conversely, suppose A and A^{-1} are both matrices with integer entries. Prove that $\det A = \pm 1$.

8. We call the vector $\mathbf{x} \in \mathbb{R}^n$ *integral* if every component x_i is an integer. Let A be a nonsingular $n \times n$ matrix with integer entries. Prove that the system of equations $A\mathbf{x} = \mathbf{b}$ has an integral solution for every integral vector $\mathbf{b} \in \mathbb{R}^n$ if and only if $\det A = \pm 1$. (Note that if A has integer entries, μ_A maps integral vectors to integral vectors. When does μ_A map the set of all integral vectors *onto* the set of all integral vectors?)

9. Prove that the exchange of any pair of rows of a matrix can be accomplished by an odd number of exchanges of adjacent pairs.

10. Suppose A is an orthogonal $n \times n$ matrix. Show that the cofactor matrix $C = \pm A$.

11. Generalizing the result of Proposition 2.4, show that $AC^{\mathsf{T}} = (\det A)I$ even if A happens to be singular. In particular, when A is singular, what can you conclude about the columns of C^{T}?

12. a. If C is the cofactor matrix of A, give a formula for $\det C$ in terms of $\det A$.

b. Let $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 0 & -1 \end{bmatrix}$. Can there be a matrix A with cofactor matrix C and $\det A = 3$? Find a matrix A with positive determinant and cofactor matrix C.

13. a. Show that if (x_1, y_1) and (x_2, y_2) are distinct points in $\mathbb{R}^2$, then the unique line passing through them is given by the equation

$$\det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = 0.$$

b. Show that if (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are noncollinear points in $\mathbb{R}^3$, then the unique plane passing through them is given by the equation

$$\det \begin{bmatrix} 1 & x & y & z \\ 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{bmatrix} = 0.$$

14. As we saw in Exercises 1.6.7 and 1.6.11, through any three noncollinear points in $\mathbb{R}^2$ there pass a unique parabola $y = ax^2 + bx + c$ and a unique circle $x^2 + y^2 + ax + by + c = 0$. (In the case of the parabola, we must also assume that no two of the points lie on a vertical line.) Given three such points (x_1, y_1), (x_2, y_2), and (x_3, y_3), show that the equation of the parabola and circle are, respectively,

$$\det \begin{bmatrix} 1 & x & x^2 & y \\ 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{bmatrix} = 0 \quad \text{and} \quad \det \begin{bmatrix} 1 & x & y & x^2+y^2 \\ 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \end{bmatrix} = 0.$$

15. (from the 1994 Putnam Exam) Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries. (*Hint:* Use Exercise 7.)

16. In this problem, let $D(\mathbf{x}, \mathbf{y})$ denote the determinant of the 2×2 matrix with rows $\mathbf{x}$ and $\mathbf{y}$. Assume the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ are pairwise linearly independent.

a. Prove that $D(\mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + D(\mathbf{v}_3, \mathbf{v}_1)\mathbf{v}_2 + D(\mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3 = \mathbf{0}$. (*Hint:* Write $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ and use Cramer's Rule to solve for the coefficients.)

b. Now suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$ and for $i = 1, 2, 3$, let ℓ_i be the line in the plane passing through $\mathbf{a}_i$ with direction vector $\mathbf{v}_i$. Prove that the three lines have a point in common if and only if

$$D(\mathbf{a}_1, \mathbf{v}_1)D(\mathbf{v}_2, \mathbf{v}_3) + D(\mathbf{a}_2, \mathbf{v}_2)D(\mathbf{v}_3, \mathbf{v}_1) + D(\mathbf{a}_3, \mathbf{v}_3)D(\mathbf{v}_1, \mathbf{v}_2) = 0.$$

(*Hint:* Use Cramer's Rule to get an equation that says that the point of intersection of ℓ_1 and ℓ_2 lies on ℓ_3.)

17. Using Exercise 16, prove that the perpendicular bisectors of the sides of a triangle have a common point. (*Hint:* If $\rho\colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation through an angle $\pi/2$ counterclockwise, show that $D(\mathbf{x}, \rho(\mathbf{y})) = \mathbf{x} \cdot \mathbf{y}$.)

3 Signed Area in $\mathbb{R}^2$ and Signed Volume in $\mathbb{R}^3$

We now turn to the geometric interpretation of the determinant. We start with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and consider the parallelogram $\mathcal{P}$ they span. The area of $\mathcal{P}$ is nonzero so long as $\mathbf{x}$ and $\mathbf{y}$ are not collinear, i.e., so long as $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent. We want to express the area of $\mathcal{P}$ in terms of the coordinates of $\mathbf{x}$ and $\mathbf{y}$. First notice that the area of the parallelogram pictured in Figure 3.1 is the same as the area of the rectangle obtained by moving the shaded triangle from the right side to the left. This rectangle has area $A = bh$, where $b = \|\mathbf{x}\|$ is the base and $h = \|\mathbf{y}\| \sin\theta$ is the height. Remembering that $\sin\theta = \cos(\frac{\pi}{2} - \theta)$ and using the fundamental formula for dot product on p. 24, we have (see Figure 3.2)

$$\|\mathbf{x}\|\|\mathbf{y}\| \sin\theta = \|\mathbf{x}\|\|\mathbf{y}\| \cos\left(\frac{\pi}{2} - \theta\right) = \rho(\mathbf{x}) \cdot \mathbf{y},$$

where $\rho(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ an angle $\pi/2$ counterclockwise. If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then we have

$$\text{area}(\mathcal{P}) = \rho(\mathbf{x}) \cdot \mathbf{y} = (-x_2, x_1) \cdot (y_1, y_2) = x_1y_2 - x_2y_1,$$

which we notice is the determinant of the 2×2 matrix with row vectors $\mathbf{x}$ and $\mathbf{y}$.

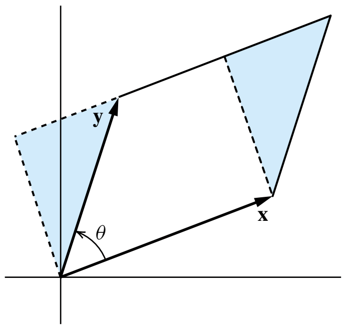

FIGURE 3.1

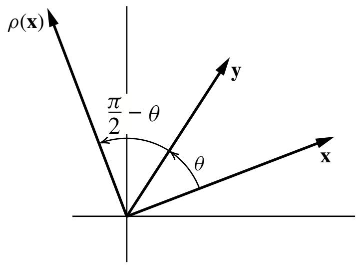

FIGURE 3.2

EXAMPLE 1

If $\mathbf{x} = (3, 1)$ and $\mathbf{y} = (4, 3)$, then the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$ is $x_1y_2 - x_2y_1 = 3 \cdot 3 - 1 \cdot 4 = 5$. On the other hand, if we interchange the two, letting $\mathbf{x} = (4, 3)$ and $\mathbf{y} = (3, 1)$, then we get $x_1y_2 - x_2y_1 = 4 \cdot 1 - 3 \cdot 3 = -5$. Certainly the parallelogram hasn't changed, nor does it make sense to have negative area. What is the explanation? In deriving our formula for the area above, we assumed $0 < \theta < \pi$; but if we must turn clockwise to get from $\mathbf{x}$ to $\mathbf{y}$, this means that θ is negative, resulting in a sign discrepancy in the area calculation. ▲

This example forces us to amend our earlier result. As indicated in Figure 3.3, we define the *signed area* of the parallelogram $\mathcal{P}$ to be the area of $\mathcal{P}$ when one turns *counterclockwise* from $\mathbf{x}$ to $\mathbf{y}$ and to be *negative* the area of $\mathcal{P}$ when one turns *clockwise* from $\mathbf{x}$ to $\mathbf{y}$. Then we have

$$\text{signed area}(\mathcal{P}) = x_1 y_2 - x_2 y_1.$$

We use $D(\mathbf{x}, \mathbf{y})$ to represent the signed area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$, in that order.

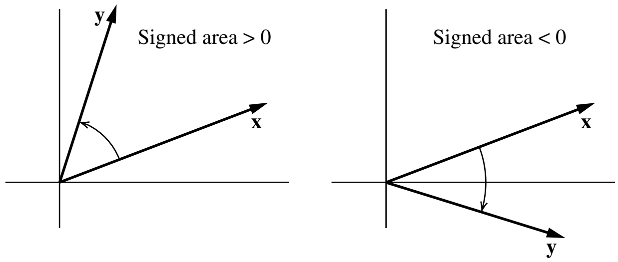

FIGURE 3.3

Next, we observe that the signed area satisfies properties **1**, **2**, **3**, and **4** of the determinant. The first is built into the definition of *signed* area. If we stretch one of the edges of the parallelogram by a factor of $c > 0$, then the area is multiplied by a factor of c. And if $c < 0$, the area is multiplied by a factor of $|c|$ and the signed area changes sign (why?). So property **2** holds. Property **3** is *Cavalieri's principle*, as illustrated in Figure 3.4: If two parallelograms have the same height and cross sections of equal lengths at corresponding heights, then they have the same area. That is, when we shear a parallelogram, we do not change its area. Property **4** is immediate because $D(\mathbf{e}_1, \mathbf{e}_2) = 1$.

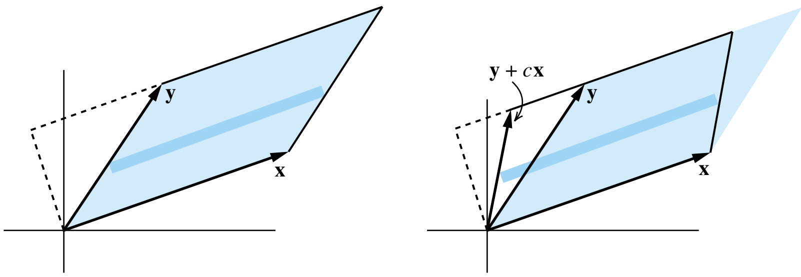

FIGURE 3.4

Interestingly, we can deduce property **3**′ from Figure 3.5: The area of parallelogram $OBCD$ ($D(\mathbf{x}+\mathbf{y}, \mathbf{z})$) is equal to the sum of the areas of parallelograms $OAED$ ($D(\mathbf{x}, \mathbf{z})$) and $ABCE$ ($D(\mathbf{y}, \mathbf{z})$). The proof of this, in turn, follows from the fact that $\triangle OAB$ is congruent to $\triangle DEC$.

Similarly, moving to three dimensions, let's consider the *signed volume* $D(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of the parallelepiped spanned by three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} \in \mathbb{R}^3$. Once again we observe that this quantity satisfies properties **1**, **2**, **3**, and **4** of the determinant. We will say in a minute how the sign is determined, but then property **1** will be evident. Properties **2** and **4** are again immediate. And Property **3** is again Cavalieri's principle, as we see in Figure 3.6. If two solids have the same height and cross sections of equal areas at corresponding heights, then they have the same volume.

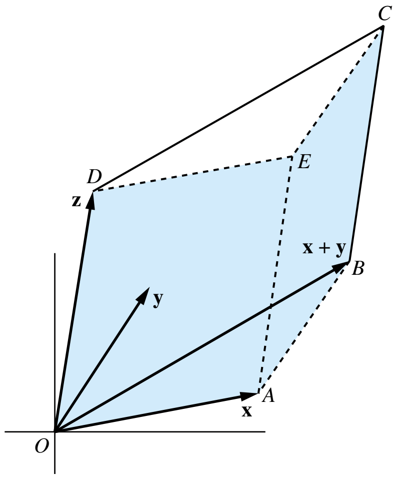

FIGURE 3.5

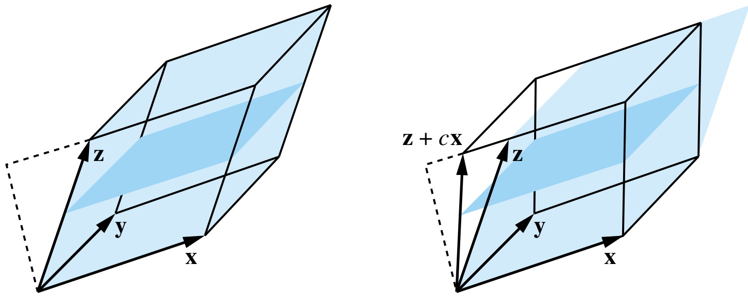

FIGURE 3.6

Here is how we decide the *sign* of the signed volume. We apply the *right-hand rule* familiar to most multivariable calculus and physics students: As shown in Figure 3.7, one lines up the fingers of one's right hand with the vector $\mathbf{x}$ and curls them toward $\mathbf{y}$. If one's thumb now is on the same side of the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{z}$ is, then the signed volume is *positive*; if one's thumb is on the opposite side, then the signed volume is *negative*. Note that this definition is exactly what it takes for signed volume in $\mathbb{R}^3$ to have property **1** of determinants.

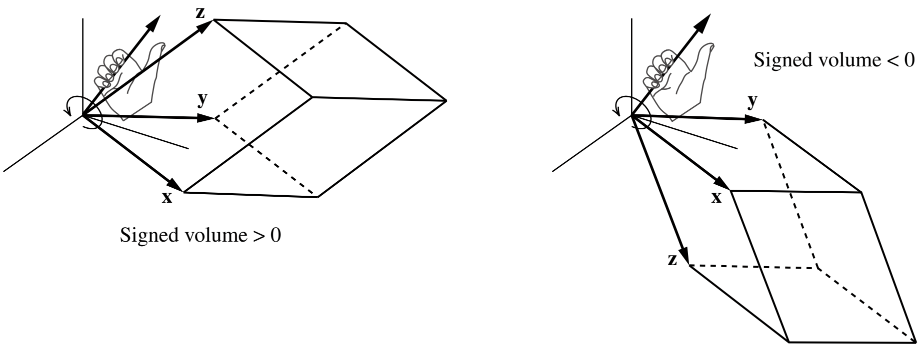

FIGURE 3.7

We leave it for the reader to explore the notion of volume and signed volume in higher dimensions, but for our purposes here, we will just say that the signed n-dimensional volume of the parallelepiped spanned by vectors $\mathbf{A}_1, \dots, \mathbf{A}_n \in \mathbb{R}^n$, denoted $D(\mathbf{A}_1, \dots, \mathbf{A}_n)$, coincides with the determinant of the matrix A with rows $\mathbf{A}_i$.

Now we close with a beautiful and important result, one that is used in establishing the powerful change-of-variables theorem in multivariable calculus.

Proposition 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\mathcal{P} \subset \mathbb{R}^n$ be the parallelepiped spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then the signed volume of the parallelepiped $T(\mathcal{P})$ is equal to the product of the signed volume of the parallelepiped $\mathcal{P}$ and* $\det T$.

Remark. By definition, $\det T$ is the signed volume of the parallelepiped spanned by the vectors $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$. This number, amazingly, gives the ratio of the signed volume of $T(\mathcal{P})$ to the signed volume of $\mathcal{P}$ for *every* parallelepiped $\mathcal{P}$, as indicated in Figure 3.8.

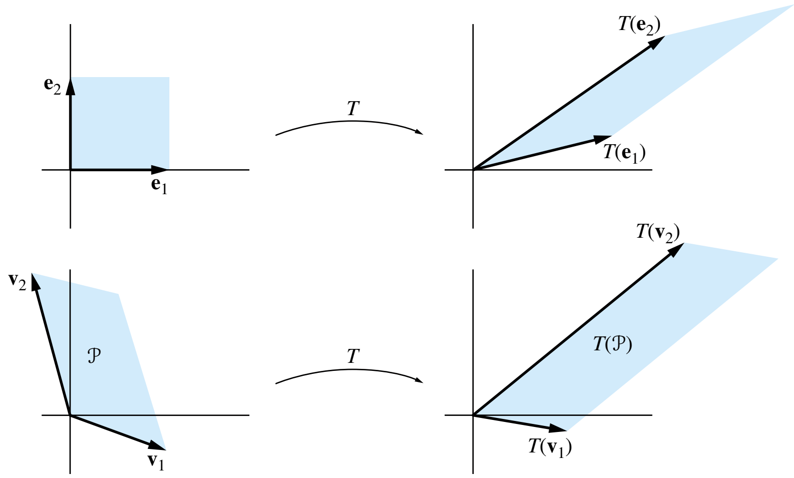

FIGURE 3.8

Proof. The signed volume of $\mathcal{P}$ is given by $D(\mathbf{v}_1, \dots, \mathbf{v}_n)$, which, by definition, is the determinant of the matrix whose *row* vectors are $\mathbf{v}_1, \dots, \mathbf{v}_n$. By Proposition 1.7, this is in turn the determinant of the matrix whose *columns* are $\mathbf{v}_1, \dots, \mathbf{v}_n$. Similarly, letting A be the standard matrix for T, the signed volume of $T(\mathcal{P})$ is given by

$$D(T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)) = \det \begin{bmatrix} | & & | \\ A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \\ | & & | \end{bmatrix} = \det \left(\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \right).$$

Using the product rule for determinants, Theorem 1.5, we infer that

$$D(T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)) = (\det A)\ D(\mathbf{v}_1, \dots, \mathbf{v}_n) = (\det T)\ D(\mathbf{v}_1, \dots, \mathbf{v}_n),$$

as required. □

Exercises 5.3

1. Find the signed area of the parallelogram formed by the following pairs of vectors in $\mathbb{R}^2$.
   *a. $\mathbf{x} = (1, 5)$, $\mathbf{y} = (2, 3)$
   b. $\mathbf{x} = (4, 3)$, $\mathbf{y} = (5, 4)$
   c. $\mathbf{x} = (2, 5)$, $\mathbf{y} = (3, 7)$

2. Find the signed volume of the parallelepiped formed by the following triples of vectors in $\mathbb{R}^3$.

*a. $\mathbf{x} = (1, 2, 1)$, $\mathbf{y} = (2, 3, 1)$, $\mathbf{z} = (-1, 0, 3)$

b. $\mathbf{x} = (1, 1, 1)$, $\mathbf{y} = (2, 3, 4)$, $\mathbf{z} = (1, 1, 5)$

c. $\mathbf{x} = (3, -1, 2)$, $\mathbf{y} = (1, 0, -3)$, $\mathbf{z} = (-2, 1, -1)$

3. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be points in $\mathbb{R}^2$. Show that the signed area of $\triangle ABC$ is given by

$$\frac{1}{2}\det\begin{bmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{bmatrix}.$$

4. Suppose A, B, and C are vertices of a triangle in $\mathbb{R}^2$, and D is a point in $\mathbb{R}^2$.

a. Use the fact that the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent to prove that we can write $D = rA + sB + tC$ for some scalars r, s, and t with $r + s + t = 1$. (Here, we are treating A, B, C, and D as vectors in $\mathbb{R}^2$.)

b. Use Exercise 3 to show that t is the ratio of the signed area of $\triangle ABD$ to the signed area of $\triangle ABC$ (and similar results hold for r and s).

5. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Define the *cross product* of $\mathbf{u}$ and $\mathbf{v}$ to be the vector

$$\mathbf{u} \times \mathbf{v} = \left(\det\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}, \det\begin{bmatrix} u_3 & u_1 \\ v_3 & v_1 \end{bmatrix}, \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right).$$

a. Prove that for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in \mathbb{R}^3$, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = D(\mathbf{w}, \mathbf{u}, \mathbf{v})$.

b. Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$.

6. Let $\mathcal{P}$ be the parallelogram spanned by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

a. By interpreting $D(\mathbf{u} \times \mathbf{v}, \mathbf{u}, \mathbf{v})$ as both a signed volume and a determinant, show that $\text{area}(\mathcal{P}) = \|\mathbf{u} \times \mathbf{v}\|$. (*Hint:* For the latter, expand in cofactors.)

b. Let $\mathcal{P}_1$ be the projection of $\mathcal{P}$ onto the x_2x_3-plane; $\mathcal{P}_2$, its projection onto the x_1x_3-plane; and $\mathcal{P}_3$, its projection onto the x_1x_2-plane. Show that

$$\big(\text{area}(\mathcal{P})\big)^2 = \big(\text{area}(\mathcal{P}_1)\big)^2 + \big(\text{area}(\mathcal{P}_2)\big)^2 + \big(\text{area}(\mathcal{P}_3)\big)^2.$$

How's that for a generalization of the Pythagorean Theorem?!

7. Let $\mathbf{a} \in \mathbb{R}^3$ be fixed. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$ (see Exercise 5).

a. Prove that T is a linear transformation.

b. Give the standard matrix A of T.

c. Explain, using part *a* of Exercise 5 and Proposition 5.2 of Chapter 2, why A is skew-symmetric.

8. Suppose a polygon in the plane has vertices $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Give a formula for its area. (*Hint:* To start, assume that the origin is inside the polygon; draw a picture.)

9. (from the 1994 Putnam Exam) Find the value of m so that the line $y = mx$ bisects the region

$$\left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 \le 1,\ x \ge 0,\ y \ge 0\right\}.$$

(*Hint:* How are ellipses, circles, and linear transformations related?)

10. Given any ellipse, show that there are infinitely many inscribed triangles of maximal area. (*Hint:* See the hint for Exercise 9.)

11. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Show that

$$\det \begin{bmatrix} \mathbf{x}\cdot\mathbf{x} & \mathbf{x}\cdot\mathbf{y} \\ \mathbf{y}\cdot\mathbf{x} & \mathbf{y}\cdot\mathbf{y} \end{bmatrix}$$

is the square of the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$.

12. Generalizing the result of Exercise 11, let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$. Show that

$$\det \begin{bmatrix} \mathbf{v}_1\cdot\mathbf{v}_1 & \mathbf{v}_1\cdot\mathbf{v}_2 & \dots & \mathbf{v}_1\cdot\mathbf{v}_k \\ \mathbf{v}_2\cdot\mathbf{v}_1 & \mathbf{v}_2\cdot\mathbf{v}_2 & \dots & \mathbf{v}_2\cdot\mathbf{v}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_k\cdot\mathbf{v}_1 & \mathbf{v}_k\cdot\mathbf{v}_2 & \dots & \mathbf{v}_k\cdot\mathbf{v}_k \end{bmatrix}$$

is the square of the volume of the k-dimensional parallelepiped spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.

HISTORICAL NOTES

Determinants first arose as an aid to solving equations. Although 2×2 determinants were implicit in the solution of a system of two linear equations in two unknowns given by Girolamo Cardano (1501–1576) in his work *Ars Magna* (1545), the Japanese mathematician Takakazu Seki (1642–1708) is usually credited with a more general study of determinants. Seki, the son of a samurai, was a self-taught mathematical prodigy who developed quite a following in seventeenth-century Japan. In 1683 he published *Method of Solving Dissimulated Problems*, in which he studied determinants of matrices at least as large as 5×5. In the same year the German mathematician Gottfried Wilhelm von Leibniz (1646–1716) wrote a letter to Guillaume de l'Hôpital (1661–1704), in which he gave the vanishing of the determinant of a 3×3 system of linear equations as the condition for the homogeneous system to have a nontrivial solution. Although Leibniz never published his work on determinants, his notes show that he understood many of their properties and uses, as well as methods for computing them.

After Seki and Leibniz, determinants found their way into the work of many mathematicians. In 1750 the Swiss mathematician Gabriel Cramer (1704–1752) published, without proof and as an appendix to a book on plane algebraic curves, what is now called Cramer's Rule. Other eighteenth-century mathematicians who studied methods for computing determinants and uses for them were Étienne Bézout (1730–1783), Alexandre Vandermonde (1735–1796), and Pierre-Simon Laplace (1749–1847), who developed the method of expansion by cofactors for computing determinants.

The French mathematician Joseph-Louis Lagrange (1736–1813) seems to have been the first to notice the relationship between determinants and volume. In 1773 he used a 3×3 determinant to compute the volume of a tetrahedron. Carl Friedrich Gauss (1777–1855) used matrices to study the properties of quadratic forms (see Section 4 of Chapter 6). Augustin Louis Cauchy (1789–1857) also studied determinants in the context of quadratic forms and is given credit for the first proof of the multiplicative properties of the determinant. Finally, three papers that Carl Gustav Jacob Jacobi (1804–1851) wrote in 1841 brought general attention to the theory of determinants. Jacobi, a brilliant German mathematician whose life was cut short by smallpox, focused his attention on determinants of matrices with functions as entries. He proved key results regarding the independence of sets of functions, inventing a particular determinant that is now called the Jacobian determinant. It plays a major role in the change-of-variables formula in multivariable calculus, generalizing Proposition 3.1.

CHAPTER 6

EIGENVALUES AND EIGENVECTORS

We suggested in Chapter 4 that a linear transformation $T: V \to V$ is best understood when there is a basis for V with respect to which the matrix of T becomes diagonal. In this chapter, we shall develop techniques for determining whether T is diagonalizable and, if so, for finding a diagonalizing basis.

There are important reasons to diagonalize a matrix. For instance, we saw a long while ago (for example, in Examples 6 through 9 in Section 6 of Chapter 1) that it is often necessary to understand and calculate (high) powers of a given square matrix. Suppose A is diagonalizable, i.e., there is an invertible matrix P so that $P^{-1}AP = \Lambda$ is diagonal. Then we have

$$A = P\Lambda P^{-1}, \quad \text{and so}$$
$$A^k = \underbrace{(P\Lambda P^{-1})(P\Lambda P^{-1})\cdots(P\Lambda P^{-1})}_{k \text{ times}} = P\Lambda^k P^{-1},$$

using associativity to regroup and cancel the $P^{-1}P$ pairs. Since Λ^k is easy to calculate, we are left with a formula for A^k that helps us understand the corresponding linear transformation *and* is easy to compute. We will see a number of applications of this principle in Section 3. Indeed, we will see that the entries of Λ tell us a lot about growth in discrete dynamical systems and whether systems approach a "steady state" in time. Further applications, to understanding conic sections and quadric surfaces and to systems of differential equations, are given in Section 4 and in Section 3 of Chapter 7, respectively.

We turn first to the matter of finding the diagonal matrix Λ if, in fact, A is diagonalizable. Then we will develop some criteria that guarantee diagonalizability.

1 The Characteristic Polynomial

Recall that a linear transformation $T: V \to V$ is *diagonalizable* if there is an (ordered) basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V so that the matrix for T with respect to that basis is diagonal. This means precisely that, for some scalars $\lambda_1, \dots, \lambda_n$, we have

$$\begin{aligned} T(\mathbf{v}_1) &= \lambda_1\mathbf{v}_1, \\ T(\mathbf{v}_2) &= \lambda_2\mathbf{v}_2, \\ &\vdots \\ T(\mathbf{v}_n) &= \lambda_n\mathbf{v}_n. \end{aligned}$$

Likewise, an $n \times n$ matrix A is diagonalizable if the associated linear transformation $\mu_A : \mathbb{R}^n \to \mathbb{R}^n$ is diagonalizable; so A is diagonalizable precisely when there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ with the property that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all $i = 1, \dots, n$. We can write these equations in matrix form:

$$A \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

Thus, if we let P be the $n \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and Λ be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then we have

$$AP = P\Lambda, \quad \text{and so} \quad P^{-1}AP = \Lambda.$$

(Of course, this all follows immediately from the change-of-basis formula, Proposition 3.2 of Chapter 4. In that context, we called P the change-of-basis matrix.)

This observation leads us to the following definition.

> ***Definition.*** Let $T: V \to V$ be a linear transformation. A *nonzero* vector $\mathbf{v} \in V$ is called an *eigenvector*[1] of T if there is a scalar λ so that $T(\mathbf{v}) = \lambda\mathbf{v}$. The scalar λ is called the associated *eigenvalue* of T.

In other words, an eigenvector of a linear transformation T is a nonzero vector that is rescaled (perhaps in the negative direction) by T. The line spanned by the vector is identical to the line spanned by its image under T.

EXAMPLE 1

Revisiting Example 8 in Section 3 of Chapter 4, we see that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of T, with associated eigenvalues 4 and 1. ▲

This definition, in turn, leads to a convenient reformulation of diagonalizability:

Proposition 1.1. *The linear transformation $T: V \to V$ is diagonalizable if and only if there is a basis for V consisting of eigenvectors of T.*

At this juncture, the obvious question to ask is how we should find eigenvectors. As a matter of convenience, we're now going to stick mostly to the more familiar matrix notation since we'll be starting with an $n \times n$ matrix most of the time anyhow. For general linear transformations, let A denote the matrix for T with respect to *some* basis. Let's start by observing that the set of eigenvectors with eigenvalue λ, together with the zero vector, forms a subspace.

---

[1]In the old days, it was called a *characteristic vector*, more or less a literal translation of the German *eigen*, which means "characteristic," "proper," or "particular."

Lemma 1.2. *Let A be an $n \times n$ matrix, and let λ be any scalar. Then*

$$\mathbf{E}(\lambda) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\} = \mathbf{N}(A - \lambda I)$$

is a subspace of $\mathbb{R}^n$. Moreover, $\mathbf{E}(\lambda) \neq \{\mathbf{0}\}$ if and only if λ is an eigenvalue, in which case we call $\mathbf{E}(\lambda)$ the λ-eigenspace *of the matrix A.*

Proof. We know that $\mathbf{N}(A - \lambda I)$ is always a subspace of $\mathbb{R}^n$. By definition, λ is an eigenvalue precisely when there is a *nonzero* vector in $\mathbf{E}(\lambda)$. □

We now come to what will be for us the main computational tool for finding eigenvalues.

Proposition 1.3. *Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A if and only if* $\det(A - \lambda I) = 0$.

Proof. From Lemma 1.2 we infer that λ is an eigenvalue if and only if the matrix $A - \lambda I$ is singular. Next we conclude from Theorem 1.2 of Chapter 5 that $A - \lambda I$ is singular precisely when $\det(A - \lambda I) = 0$. Putting the two statements together, we obtain the result. □

Once we use this criterion to find the eigenvalues λ, it is an easy matter to find the corresponding eigenvectors merely by finding $\mathbf{N}(A - \lambda I)$.

EXAMPLE 2

Let's find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}.$$

We start by calculating

$$\det(A - tI) = \det \begin{bmatrix} 3-t & 1 \\ -3 & 7-t \end{bmatrix} = (3-t)(7-t) - (1)(-3) = t^2 - 10t + 24.$$

Since $t^2 - 10t + 24 = (t-4)(t-6) = 0$ when $t = 4$ or $t = 6$, these are our two eigenvalues. We now proceed to find the corresponding eigenspaces.

$\mathbf{E}(4)$**:** We see that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{gives a basis for} \quad \mathbf{N}(A - 4I) = \mathbf{N}\left(\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\right).$$

$\mathbf{E}(6)$**:** We see that

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{gives a basis for} \quad \mathbf{N}(A - 6I) = \mathbf{N}\left(\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\right).$$

Since we observe that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, the matrix A is diagonalizable. Indeed, as the reader can check, if we take $P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, then

$$P^{-1}AP = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix},$$

as should be the case. ▲

EXAMPLE 3

Let's find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}.$$

We begin by computing

$$\det(A - tI) = \det \begin{bmatrix} 1-t & 2 & 1 \\ 0 & 1-t & 0 \\ 1 & 3 & 1-t \end{bmatrix}$$

(expanding in cofactors along the second row)

$$= (1-t)\big((1-t)(1-t) - 1\big) = (1-t)(t^2 - 2t) = -t(t-1)(t-2).$$

Thus, the eigenvalues of A are $\lambda = 0$, 1, and 2. We next find the respective eigenspaces.

E(0)**:** We see that

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives a basis for} \quad \mathbf{N}(A - 0I) = \mathbf{N}\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}\right)$$

$$= \mathbf{N}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

E(1)**:** We see that

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad \text{gives a basis for} \quad \mathbf{N}(A - 1I) = \mathbf{N}\left(\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\right)$$

$$= \mathbf{N}\left(\begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\right).$$

E(2)**:** We see that

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives a basis for} \quad \mathbf{N}(A - 2I) = \mathbf{N}\left(\begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 3 & -1 \end{bmatrix}\right)$$

$$= \mathbf{N}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

Once again, A is diagonalizable. As the reader can check, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and therefore gives a basis for $\mathbb{R}^3$. Just to be sure, we let

$$P = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix};$$

then

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

as we expected. ▲

There is a built-in check here for the eigenvalues. If λ is truly to be an eigenvalue of A, we *must* find a nonzero vector in $\mathbf{N}(A - \lambda I)$. If we do not, then λ cannot be an eigenvalue.

EXAMPLE 4

Let's find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

As usual, we calculate

$$\det(A - tI) = \det \begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix} = t^2 + 1.$$

Since $t^2 + 1 \geq 1$ for all real numbers t, there is no *real* number λ so that $\det(A - \lambda I) = 0$. Thus, the matrix A has no *real* eigenvalue. Nevertheless, it is still interesting to allow eigenvalues to be complex numbers (and, then, eigenvectors to be vectors with complex entries). We will study this in greater detail in Section 1 of Chapter 7.

We recognize A as the matrix giving rotation of $\mathbb{R}^2$ through an angle of $\pi/2$. Thus, it is clear on geometric grounds that this matrix has no (real) eigenvector: For any nonzero vector $\mathbf{x}$, the vector $A\mathbf{x}$ makes a right angle with $\mathbf{x}$ and is therefore not a scalar multiple of $\mathbf{x}$.[2] ▲

It is evident that we are going to find the eigenvalues of a matrix A by finding the (real) roots of the polynomial $\det(A - tI)$. This leads us to make our next definition.

Definition. Let A be a square matrix. Then $p(t) = p_A(t) = \det(A - tI)$ is called the *characteristic polynomial* of A.[3]

[2] On the other hand, from the complex perspective, we note that $\pm i$ are the (complex) eigenvalues of A, and multiplying a complex number by i has the effect of rotating it an angle of $\pi/2$. The reader can check that the complex eigenvectors of this matrix are $(1, \pm i)$.

[3] That the characteristic polynomial of an $n \times n$ matrix is in fact a polynomial of degree n seems pretty evident from examples, but the fastidious reader can establish this by expanding in cofactors.

We can restate Proposition 1.3 by saying that the eigenvalues of A are the real roots of the characteristic polynomial $p_A(t)$. As in Example 4, this polynomial may sometimes have complex roots; we will abuse language by calling these roots *complex eigenvalues*. See Section 1 of Chapter 7 for a more detailed discussion of this situation.

Lemma 1.4. *If A and B are similar matrices, then $p_A(t) = p_B(t)$.*

Proof. Suppose $B = P^{-1}AP$. Then

$$p_B(t) = \det(B - tI) = \det(P^{-1}AP - tI) = \det\left(P^{-1}(A - tI)P\right) = \det(A - tI) = p_A(t),$$

by virtue of the product rule for determinants, Theorem 1.5 of Chapter 5. □

As a consequence, if V is a finite-dimensional vector space and $T : V \to V$ is a linear transformation, then we can define the characteristic polynomial of T to be that of the matrix A for T with respect to *any* basis for V. By the change-of-basis formula, Proposition 3.2 of Chapter 4, and Lemma 1.4, we'll get the same answer no matter what basis we choose.

Remark. In order to determine the eigenvalues of a matrix, we must find the roots of its characteristic polynomial. In real-world applications (where the matrices tend to get quite large), one might do this numerically (e.g., using Newton's method). However, there are more sophisticated methods for finding the eigenvalues without even calculating the characteristic polynomial; a powerful such method is based on the QR decomposition of a matrix. The interested reader should consult Strang's books or Wilkinson for more details.

For the lion's share of the matrices that we shall encounter here, the eigenvalues will be integers, and so we take this opportunity to remind you of a shortcut from high school algebra.

Proposition 1.5 (Rational Roots Test). *Let $p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be a polynomial with* integer *coefficients. If $t = r/s$ is a rational root (in lowest terms) of $p(t)$, then r must be a factor of a_0 and s must be a factor of a_n.*[4]

In particular, when the leading coefficient a_n is ± 1, as is always the case with the characteristic polynomial, any rational root must in fact be an integer that divides a_0. So, in practice, we test the various factors of a_0 (being careful to try both positive and negative factors). In our case, $a_0 = p(0) = \det A$, so, starting with a matrix A with all integer entries, only the factors of the integer $\det A$ can be integer eigenvalues. Once we find one root λ, we can divide $p(t)$ by $t - \lambda$ to obtain a polynomial of smaller degree.

EXAMPLE 5

The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & 1 & 4 \\ 2 & -2 & 1 \end{bmatrix}$$

is $p(t) = -t^3 + 6t^2 - 11t + 6$. The factors of 6 are $\pm 1, \pm 2, \pm 3$, and ± 6. Since $p(1) = 0$, we know that 1 is a root (so we were lucky!). Now,

$$\frac{-p(t)}{t - 1} = t^2 - 5t + 6 = (t - 2)(t - 3),$$

and we have succeeded in finding all three eigenvalues of A, namely, $\lambda = 1, 2$, and 3. ▲

[4]We do not include a proof here, but you can find one in most abstract algebra texts. For obvious reasons, we recommend Shifrin's *Abstract Algebra: A Geometric Approach*, p. 105.

Remark. It might be nice to have a few shortcuts for calculating the characteristic polynomial of small matrices. For 2×2 matrices, it's quite easy:

$$\det\begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} = (a-t)(d-t) - bc = t^2 - \boxed{(a+d)}\,t + \boxed{(ad-bc)}$$
$$= t^2 - \boxed{\operatorname{tr} A}\,t + \boxed{\det A}.$$

(Recall that the *trace* of a matrix A, denoted $\operatorname{tr} A$, is the sum of its diagonal entries.) For 3×3 matrices, it's a bit more involved:

$$\det\begin{bmatrix} a_{11}-t & a_{12} & a_{13} \\ a_{21} & a_{22}-t & a_{23} \\ a_{31} & a_{32} & a_{33}-t \end{bmatrix} = -t^3 + \boxed{(a_{11}+a_{22}+a_{33})}\,t^2$$
$$- \boxed{\big((a_{11}a_{22} - a_{12}a_{21}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{22}a_{33} - a_{23}a_{32})\big)}\,t$$
$$+ \boxed{(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33})}$$
$$= -t^3 + \boxed{\operatorname{tr} A}\,t^2 - \boxed{\mathsf{Fred}}\,t + \boxed{\det A}.$$

The coefficient of t in the characteristic polynomial has no standard name, and so it was that one of the authors' students christened the expression Fred years ago. Nevertheless, we see that Fred is the sum of the cofactors C_{11}, C_{22}, and C_{33}, i.e., the sum of the determinants of the (three) 2×2 submatrices formed by deleting identical rows and columns from A.

In general, the characteristic polynomial $p(t)$ of an $n \times n$ matrix A is always of the form

$$p(t) = (-1)^n t^n + (-1)^{n-1}\boxed{\operatorname{tr} A}\,t^{n-1} + (-1)^{n-2}\boxed{\mathsf{Fred}}\,t^{n-2} + \cdots + \boxed{\det A}.$$

Note that the constant coefficient is always $\det A$ (with no minus signs) because $p(0) = \det(A - 0I) = \det A$.

In the long run, these formulas notwithstanding, it's sometimes best to calculate the characteristic polynomial of 3×3 matrices by expansion in cofactors. If one is both attentive and fortunate, this may save the trouble of factoring the polynomial.

EXAMPLE 6

Let's find the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

We calculate the determinant by expanding in cofactors along the first row:

$$\det\begin{bmatrix} 2-t & 0 & 0 \\ 1 & 2-t & 1 \\ 0 & 1 & 2-t \end{bmatrix} = (2-t)\det\begin{bmatrix} 2-t & 1 \\ 1 & 2-t \end{bmatrix}$$
$$= (2-t)\big((2-t)^2 - 1\big) = (2-t)(t^2 - 4t + 3)$$
$$= (2-t)(t-3)(t-1).$$

But that was too easy. Let's try the characteristic polynomial of

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Again, we expand in cofactors along the first row:

$$\begin{aligned} \det \begin{bmatrix} 2-t & 0 & 1 \\ 1 & 3-t & 1 \\ 1 & 1 & 2-t \end{bmatrix} &= (2-t)\det \begin{bmatrix} 3-t & 1 \\ 1 & 2-t \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 3-t \\ 1 & 1 \end{bmatrix} \\ &= (2-t)\big((3-t)(2-t)-1\big) + \big(1-(3-t)\big) \\ &= (2-t)(t^2-5t+5)-(2-t) = (2-t)(t^2-5t+4) \\ &= (2-t)(t-1)(t-4). \end{aligned}$$

OK, perhaps we were a bit lucky there, too. ▲

Exercises 6.1

1. Find the eigenvalues and eigenvectors of the following matrices.

*a. $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 10 & -6 \\ 18 & -11 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

*f. $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

*g. $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

h. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

i. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix}$

*j. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

k. $\begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

l. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

m. $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

*n. $\begin{bmatrix} 1 & -6 & 4 \\ -2 & -4 & 5 \\ -2 & -6 & 7 \end{bmatrix}$

o. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

p. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. Show that 0 is an eigenvalue of A if and only if A is singular.

3. Show that the eigenvalues of an upper (or lower) triangular matrix are its diagonal entries.

4. What are the eigenvalues and eigenvectors of a projection? a reflection?

5. Show that if λ is an eigenvalue of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and either $b \neq 0$ or $\lambda \neq a$, then $\begin{bmatrix} b \\ \lambda - a \end{bmatrix}$ is a corresponding eigenvector.

6. Suppose A is nonsingular. Prove that the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

7. Suppose $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ.

a. Prove that for any positive integer n, $\mathbf{x}$ is an eigenvector of A^n with corresponding eigenvalue λ^n. (If you know mathematical induction, this would be a good place to use it.)

b. Prove or give a counterexample: $\mathbf{x}$ is an eigenvector of $A + I$.

c. If $\mathbf{x}$ is an eigenvector of B with corresponding eigenvalue μ, prove or give a counterexample: $\mathbf{x}$ is an eigenvector of $A + B$ with corresponding eigenvalue $\lambda + \mu$.

d. Prove or give a counterexample: If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A + B$.

8. Prove or give a counterexample: If A and B have the same characteristic polynomial, then A and B are similar.

9. Suppose A is a square matrix. Suppose $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, and $\mathbf{y}$ is an eigenvector of A^T with corresponding eigenvalue μ. Show that if $\lambda \neq \mu$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

10. Prove or give a counterexample:

a. A and A^T have the same eigenvalues.

b. A and A^T have the same eigenvectors.

11. Show that the product of the roots (real and complex) of the characteristic polynomial of A is equal to $\det A$. (*Hint:* If $\lambda_1, \dots, \lambda_n$ are the roots, show that $p(t) = \pm(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$.)

12. Consider the linear transformation $T : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ defined by $T(X) = X^\mathsf{T}$. Find its eigenvalues and the corresponding eigenspaces. (*Hint:* Consider the equation $X^\mathsf{T} = \lambda X$.)

13. In each of the following cases, find the eigenvalues and eigenvectors of the linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$.

a. $T(p)(t) = p'(t)$

*b. $T(p)(t) = tp'(t)$

c. $T(p)(t) = \int_0^t p'(u)\,du$

d. $T(p)(t) = t^2 p''(t) - tp'(t)$

14. Suppose all the entries of the matrix $B = \left[b_{ij}\right]$ are positive and $\sum_{j=1}^{n} b_{ij} = 1$ for all $i = 1, \dots, n$. Show that, up to scalar multiples, $(1, 1, \dots, 1)$ is the *unique* eigenvector of B with eigenvalue 1. (*Hint:* Let $\mathbf{x} = (x_1, \dots, x_n)$ be an eigenvector; if $|x_k| \geq |x_i|$ for all $i = 1, \dots, n$, then look carefully at the k^{th} coordinate of $B\mathbf{x} - \mathbf{x}$.)

15. *a. Let $V = \mathcal{C}^1(\mathcal{I})$ be the vector space of continuously differentiable functions on the open interval $\mathcal{I} = (0, 1)$. Define $T: V \to V$ by $T(f)(t) = tf'(t)$. Prove that every real number is an eigenvalue of T and find the corresponding eigenvectors.

b. Let $V = \{f \in \mathcal{C}^0(\mathbb{R}) : \lim_{t\to-\infty} f(t)|t|^n = 0$ for all positive integers $n\}$. (Why is V a vector space?) Define $T: V \to V$ by $T(f)(t) = \int_{-\infty}^{t} f(s)\,ds$. (If $f \in V$, why is $T(f) \in V$?) Find the eigenvalues and eigenvectors of T.

16. Let A and B be $n \times n$ matrices.

a. Suppose A (or B) is nonsingular. Prove that the characteristic polynomials of AB and BA are equal.

*b. (more challenging) Prove the result of part *a* when both A and B are singular.

2 Diagonalizability

Judging by the examples in the previous section, it seems to be the case that when an $n \times n$ matrix (or linear transformation) has n *distinct* eigenvalues, the corresponding eigenvectors form a linearly independent set and will therefore give a “diagonalizing basis.” Let’s begin by proving a slightly stronger statement.

Theorem 2.1. *Let $T: V \to V$ be a linear transformation. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ are eigenvectors of T with distinct corresponding eigenvalues $\lambda_1, \dots, \lambda_k$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set of vectors.*

Proof. Let m be the largest number between 1 and k (inclusive) so that $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent. We want to see that $m = k$. By way of contradiction, suppose $m < k$. Then we know that $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent and $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}\}$ is linearly dependent. It follows from Proposition 3.2 of Chapter 3 that $\mathbf{v}_{m+1} = c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m$ for some scalars $c_1, \dots, c_m$. Then (using repeatedly the fact that $T(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$)

$$\begin{aligned}\mathbf{0} &= (T - \lambda_{m+1}I)\mathbf{v}_{m+1} = (T - \lambda_{m+1}I)(c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m)\\ &= c_1(\lambda_1 - \lambda_{m+1})\mathbf{v}_1 + \dots + c_m(\lambda_m - \lambda_{m+1})\mathbf{v}_m.\end{aligned}$$

Since $\lambda_i - \lambda_{m+1} \neq 0$ for $i = 1, \dots, m$, and since $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent, the only possibility is that $c_1 = \dots = c_m = 0$, contradicting the fact that $\mathbf{v}_{m+1} \neq \mathbf{0}$ (by the very definition of eigenvector). Thus, it cannot happen that $m < k$, and the proof is complete. □

Remark. What is underlying this formal argument is the observation that if $\mathbf{v} \in \mathbf{E}(\lambda) \cap \mathbf{E}(\mu)$, then $T(\mathbf{v}) = \lambda\mathbf{v}$ and $T(\mathbf{v}) = \mu\mathbf{v}$. Hence, if $\lambda \neq \mu$, then $\mathbf{v} = \mathbf{0}$. That is, if $\lambda \neq \mu$, we have $\mathbf{E}(\lambda) \cap \mathbf{E}(\mu) = \{\mathbf{0}\}$.

We now arrive at our first result that gives a sufficient condition for a linear transformation to be diagonalizable. (Note that we insert the requirement that the eigenvalues be real numbers; we will discuss the situation with complex eigenvalues later.)

Corollary 2.2. *Suppose V is an n-dimensional vector space and $T: V \to V$ has n distinct (real) eigenvalues. Then T is diagonalizable.*

Proof. The set of the n corresponding eigenvectors will be linearly independent and will hence give a basis for V. The matrix for T with respect to a basis of eigenvectors is always diagonal. □

Remark. Of course, there are many diagonalizable (indeed, diagonal) matrices with repeated eigenvalues. Certainly the identity matrix and the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

are diagonal, and yet they fail to have distinct eigenvalues.

We spend the rest of this section discussing the two ways in which the hypotheses of Corollary 2.2 can fail: The characteristic polynomial may have complex roots or it may have repeated roots.

EXAMPLE 1

Consider the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The reader may well recall from Chapter 4 that the linear transformation $\mu_A : \mathbb{R}^2 \to \mathbb{R}^2$ rotates the plane through an angle of $\pi/4$. Now, what are the eigenvalues of A? The characteristic polynomial is

$$p(t) = t^2 - (\operatorname{tr} A)t + \det A = t^2 - \sqrt{2}t + 1,$$

whose roots are (by the quadratic formula)

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{1 \pm i}{\sqrt{2}}.$$

After a bit of thought, it should come as no surprise that A has no (real) eigenvector, as there can be no line through the origin that is unchanged after a rotation. We leave it to the reader to calculate the (complex) eigenvectors in Exercise 8. ▲

We have seen that when the characteristic polynomial has distinct (real) roots, we get a one-dimensional eigenspace for each. What happens if the characteristic polynomial has some repeated roots?

EXAMPLE 2

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Its characteristic polynomial is $p(t) = t^2 - 4t + 4 = (t-2)^2$, so 2 is a repeated eigenvalue. Now let's find the corresponding eigenvectors:

$$\mathbf{N}(A - 2I) = \mathbf{N}\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right) = \mathbf{N}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)$$

is one-dimensional, with basis

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

It follows that A cannot be diagonalized: Since this is (up to scalar multiples) the only eigenvector in town, there can be no basis of eigenvectors. (See also Exercise 7.) ▲

EXAMPLE 3

By applying Proposition 1.3 of Chapter 5, we see that both the matrices

$$A = \left[\begin{array}{cc|c} 2 & 0 & \\ 0 & 2 & O \\ \hline & & \begin{matrix}3 & 1\\ 0 & 3\end{matrix} \\ O & & \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 2 & 1 & \\ 0 & 2 & O \\ \hline & & \begin{matrix}3 & 0\\ 0 & 3\end{matrix} \\ O & & \end{array}\right]$$

have the characteristic polynomial $p(t) = (t-2)^2(t-3)^2$. For A, there are two linearly independent eigenvectors with eigenvalue 2 but only one linearly independent eigenvector with eigenvalue 3. For B, there are two linearly independent eigenvectors with eigenvalue 3 but only one linearly independent eigenvector with eigenvalue 2. As a result, neither matrix can be diagonalized. ▲

It would be convenient to have a bit of terminology here.

Definition. Let λ be an eigenvalue of a linear transformation. The *algebraic multiplicity* of λ is its multiplicity as a root of the characteristic polynomial $p(t)$, i.e., the highest power of $t - \lambda$ dividing $p(t)$. The *geometric multiplicity* of λ is the dimension of the λ-eigenspace $\mathbf{E}(\lambda)$.

EXAMPLE 4

For the matrices in Example 3, both the eigenvalues 2 and 3 have algebraic multiplicity 2. For matrix A, the eigenvalue 2 has geometric multiplicity 2 and the eigenvalue 3 has geometric multiplicity 1; for matrix B, the eigenvalue 2 has geometric multiplicity 1 and the eigenvalue 3 has geometric multiplicity 2. ▲

From the examples we've seen, it seems quite plausible that the geometric multiplicity of an eigenvalue can be no larger than its algebraic multiplicity, but we stop to give a proof.

Proposition 2.3. *Let λ be an eigenvalue of algebraic multiplicity m and geometric multiplicity d. Then $1 \le d \le m$.*

Proof. Suppose λ is an eigenvalue of the linear transformation T. Then $d = \dim \mathbf{E}(\lambda) \geq 1$ by definition. Now, choose a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ for $\mathbf{E}(\lambda)$ and extend it to a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V. Then the matrix for T with respect to the basis $\mathcal{B}$ is of the form

$$A = \left[\begin{array}{c|c} \lambda I_d & B \\ \hline O & C \end{array}\right],$$

and so, by part a of Exercise 5.1.9, the characteristic polynomial

$$p_A(t) = \det(A - tI) = \det\big((\lambda - t)I_d\big)\det(C - tI) = (\lambda - t)^d \det(C - tI).$$

Since the characteristic polynomial does not depend on the basis, and since $(t - \lambda)^m$ is the largest power of $t - \lambda$ dividing the characteristic polynomial, it follows that $d \le m$. □

We are now able to give a necessary and sufficient criterion for a linear transformation to be diagonalizable. Based on our experience with examples, it should come as no great surprise.

Theorem 2.4. *Let $T: V \to V$ be a linear transformation. Let its distinct eigenvalues be $\lambda_1, \dots, \lambda_k$ and assume these are all real numbers. Then T is diagonalizable if and only if the geometric multiplicity, d_i, of each λ_i equals its algebraic multiplicity, m_i.*

Proof. Let V be an n-dimensional vector space. Then the characteristic polynomial of T has degree n, and we have

$$p(t) = \pm(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2}\cdots(t - \lambda_k)^{m_k};$$

therefore,

$$n = \sum_{i=1}^{k} m_i.$$

Now, suppose T is diagonalizable. Then there is a basis $\mathcal{B}$ consisting of eigenvectors. At most d_i of these basis vectors lie in $\mathbf{E}(\lambda_i)$, and so $n \le \sum_{i=1}^{k} d_i$. On the other hand, by Proposition 2.3, we know that $d_i \le m_i$ for $i = 1, \dots, k$. Putting these together, we have

$$n \le \sum_{i=1}^{k} d_i \le \sum_{i=1}^{k} m_i = n.$$

Thus, there must be equality at each stage here, which implies that $d_i = m_i$ for all $i = 1, \dots, k$.

Conversely, suppose $d_i = m_i$ for $i = 1, \dots, k$. If we choose a basis $\mathcal{B}_i$ for each eigenspace $\mathbf{E}(\lambda_i)$ and let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$, then we assert that $\mathcal{B}$ is a basis for V. There are n vectors in $\mathcal{B}$, so we need only check that the set of vectors is linearly independent. This is a generalization of the argument of Theorem 2.1, and we leave it to Exercise 20. □

EXAMPLE 5

The matrices

$$A = \begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

both have characteristic polynomial $p(t) = -(t-1)^2(t-2)$. That is, the eigenvalue 1 has algebraic multiplicity 2 and the eigenvalue 2 has algebraic multiplicity 1. To decide whether the matrices are diagonalizable, we need to know the geometric multiplicity of the eigenvalue 1. Well,

$$A - I = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 1 and so $\dim \mathbf{E}_A(1) = 2$. We infer from Theorem 2.4 that A is diagonalizable. Indeed, as the reader can check, a diagonalizing basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

On the other hand,

$$B - I = \begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2 and so $\dim \mathbf{E}_B(1) = 1$. Since the eigenvalue 1 has geometric multiplicity 1, it follows from Theorem 2.4 that B is *not* diagonalizable. ▲

In the next section we will see the power of diagonalizing matrices in several applications.

Exercises 6.2

*__1.__ Decide whether each of the matrices in Exercise 6.1.1 is diagonalizable. Give your reasoning.

2. Prove or give a counterexample:
 a. If A is an $n \times n$ matrix with n distinct (real) eigenvalues, then A is diagonalizable.
 b. If A is diagonalizable and $AB = BA$, then B is diagonalizable.
 c. If there is an invertible matrix P so that $A = P^{-1}BP$, then A and B have the same eigenvalues.
 d. If A and B have the same eigenvalues, then there is an invertible matrix P so that $A = P^{-1}BP$.
 e. There is no real 2×2 matrix A satisfying $A^2 = -I$.
 f. If A and B are diagonalizable and have the same eigenvalues (with the same algebraic multiplicities), then there is an invertible matrix P so that $A = P^{-1}BP$.

3. Suppose A is a 2×2 matrix whose eigenvalues are integers. If $\det A = 120$, explain why A must be diagonalizable.

*__4.__ Consider the differentiation operator $D\colon \mathcal{P}_k \to \mathcal{P}_k$. Is it diagonalizable?

5. Let $f_1(t) = e^t$, $f_2(t) = te^t$, $f_3(t) = t^2e^t$, and let $V = \text{Span}(f_1, f_2, f_3) \subset \mathcal{C}^\infty(\mathbb{R})$. Let $T\colon V \to V$ be given by $T(f) = f'' - 2f' + f$. Decide whether T is diagonalizable.

6. Is the linear transformation $T\colon \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ defined by $T(X) = X^{\mathsf{T}}$ diagonalizable? (See Exercise 6.1.12; Exercises 2.5.22 and 3.6.8 may also be relevant.)

*7. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. We saw in Example 2 that A has repeated eigenvalue 2 and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans $\mathbf{E}(2)$.

a. Calculate $(A - 2I)^2$.

b. Solve $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$. Explain how we know *a priori* that this equation has a solution.

c. Give the matrix for A with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

This is the closest to diagonal one can get and is called the *Jordan canonical form* of A. We'll explore this thoroughly in Section 1 of Chapter 7.

*8. Calculate the (complex) eigenvalues and (complex) eigenvectors of the rotation matrix $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

9. Prove that if λ is an eigenvalue of A with geometric multiplicity d, then λ is an eigenvalue of A^T with geometric multiplicity d. (*Hint:* Use Theorem 4.6 of Chapter 3.)

10. Here you are asked to complete a different proof of Theorem 2.1.

a. Show first that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, and apply $T - \lambda_2 I$ to this equation. Use the fact that $\lambda_1 \neq \lambda_2$ to deduce $c_1 = 0$ and, hence, that $c_2 = 0$.

b. Show next that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (Proceed as in part *a*, applying $T - \lambda_3 I$ to the equation.)

c. Continue.

11. Suppose A is an $n \times n$ matrix with the property that $A^2 = A$.

a. Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

b. Prove that A is diagonalizable. (*Hint:* See Exercise 3.2.13.)

12. Suppose A is an $n \times n$ matrix with the property that $A^2 = I$.

a. Show that if λ is an eigenvalue of A, then $\lambda = 1$ or $\lambda = -1$.

b. Prove that

$$\mathbf{E}(1) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \tfrac{1}{2}(\mathbf{u} + A\mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{R}^n\} \quad \text{and}$$
$$\mathbf{E}(-1) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \tfrac{1}{2}(\mathbf{u} - A\mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{R}^n\}.$$

c. Prove that $\mathbf{E}(1) + \mathbf{E}(-1) = \mathbb{R}^n$ and deduce that A is diagonalizable.

(For an application, see Exercise 6 and Exercise 3.6.8.)

13. Suppose A is diagonalizable, and let $p_A(t)$ denote the characteristic polynomial of A. Show that $p_A(A) = \mathrm{O}$. This result is a special case of the Cayley-Hamilton Theorem.

14. This problem gives a generalization of Exercises 11 and 12 for those readers who recall the technique of partial fraction decomposition from their calculus class. Suppose $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ are *distinct* and $f(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_k)$. Suppose A is an $n \times n$ matrix that satisfies the equation $f(A) = \mathrm{O}$. We want to show that A is diagonalizable.

a. By considering the partial fractions decomposition

$$\frac{1}{f(t)} = \frac{c_1}{t - \lambda_1} + \cdots + \frac{c_k}{t - \lambda_k},$$

show that there are polynomials $f_1(t), \dots, f_k(t)$ satisfying $1 = \sum_{j=1}^{k} f_j(t)$ and

$(t - \lambda_j) f_j(t) = c_j f(t)$ for $j = 1, \dots, k$. Conclude that we can write $I = \sum_{j=1}^{k} f_j(A)$, where $(A - \lambda_j I) f_j(A) = O$ for $j = 1, \dots, k$.

b. Show that every $\mathbf{x} \in \mathbb{R}^n$ can be decomposed as a sum of vectors $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$, where $\mathbf{x}_j \in \mathbf{E}(\lambda_j)$ for $j = 1, \dots, k$. (*Hint:* Use the result of part *a*.)

c. Deduce that A is diagonalizable.

15. Let A be an $n \times n$ matrix all of whose eigenvalues are real numbers. Prove that there is a basis for $\mathbb{R}^n$ with respect to which the matrix for A becomes upper triangular. (*Hint:* Consider a basis $\{\mathbf{v}_1, \mathbf{v}_2', \dots, \mathbf{v}_n'\}$, where $\mathbf{v}_1$ is an eigenvector. Then repeat the argument with a smaller matrix.)

16. Let A be an orthogonal 3×3 matrix.

a. Prove that the characteristic polynomial $p(t)$ has a real root.

b. Prove that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^3$ and deduce that only 1 and -1 can be (real) eigenvalues of A.

c. Prove that if $\det A = 1$, then 1 must be an eigenvalue of A.

d. Prove that if $\det A = 1$ and $A \neq I$, then $\mu_A : \mathbb{R}^3 \to \mathbb{R}^3$ is given by rotation through some angle θ about some axis. (*Hint:* First show $\dim \mathbf{E}(1) = 1$. Then show that μ_A maps $\mathbf{E}(1)^\perp$ to itself and use Exercise 2.5.19.)

e. (See the remark on p. 218.) Prove that the composition of rotations in $\mathbb{R}^3$ is again a rotation.

17. We say an $n \times n$ matrix N is *nilpotent* if $N^r = O$ for some positive integer r.

a. Show that 0 is the only eigenvalue of N.

b. Suppose $N^n = O$ and $N^{n-1} \neq O$. Prove that there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ with respect to which the matrix for N becomes

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

(*Hint:* Choose $\mathbf{v}_1 \neq \mathbf{0}$ in $\mathbf{C}(N^{n-1})$, and then define $\mathbf{v}_2, \dots, \mathbf{v}_n$ appropriately to end up with this matrix representation. To argue that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, you might want to mimic the proof of Theorem 2.1.)

18. Suppose $T : V \to V$ is a linear transformation. Suppose T is diagonalizable (i.e., there is a basis for V consisting of eigenvectors of T). Suppose, moreover, that there is a subspace $W \subset V$ with the property that $T(W) \subset W$. Prove that there is a basis for W consisting of eigenvectors of T. (*Hint:* Using Exercise 3.4.17, concoct a basis for V by starting with a basis for W. Consider the matrix for T with respect to this basis. What is its characteristic polynomial?)

19. Suppose A and B are $n \times n$ matrices.

a. Suppose that both A and B are diagonalizable and that they have the same eigenvectors. Prove that $AB = BA$.

b. Suppose A has n distinct eigenvalues and $AB = BA$. Prove that every eigenvector of A is also an eigenvector of B. Conclude that B is diagonalizable. (Query: Need every eigenvector of B be an eigenvector of A?)

c. Suppose A and B are diagonalizable and $AB = BA$. Prove that A and B are simultaneously diagonalizable; i.e., there is a nonsingular matrix P so that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal. (*Hint:* If $\mathbf{E}(\lambda)$ is the λ-eigenspace for A, show that if $\mathbf{v} \in \mathbf{E}(\lambda)$, then $B(\mathbf{v}) \in \mathbf{E}(\lambda)$. Now use Exercise 18.)

20. a. Let λ and μ be distinct eigenvalues of a linear transformation. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset \mathbf{E}(\lambda)$ is linearly independent and $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\} \subset \mathbf{E}(\mu)$ is linearly independent. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ is linearly independent.

b. More generally, if $\lambda_1, \dots, \lambda_k$ are distinct and $\{\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{d_i}^{(i)}\} \subset \mathbf{E}(\lambda_i)$ is linearly independent for $i = 1, \dots, k$, prove that $\{\mathbf{v}_j^{(i)} : i = 1, \dots, k,\ j = 1, \dots, d_i\}$ is linearly independent.

3 Applications

Suppose A is a diagonalizable matrix. Then there is a nonsingular matrix P so that

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

where the diagonal entries of Λ are the eigenvalues $\lambda_1, \dots, \lambda_n$ of A. Then it is easy to use this to calculate the powers of A:

$$\begin{aligned} A &= P\Lambda P^{-1} \\ A^2 &= (P\Lambda P^{-1})^2 = (P\Lambda P^{-1})(P\Lambda P^{-1}) = P\Lambda(P^{-1}P)\Lambda P^{-1} = P\Lambda^2 P^{-1} \\ A^3 &= A^2 A = (P\Lambda^2 P^{-1})(P\Lambda P^{-1}) = P\Lambda^2(P^{-1}P)\Lambda P^{-1} = P\Lambda^3 P^{-1} \\ &\ \vdots \\ A^k &= P\Lambda^k P^{-1}. \end{aligned}$$

We saw in Section 6 of Chapter 1 a number of examples of difference equations, which are solved by finding the powers of a matrix. We are now equipped to tackle these problems.

EXAMPLE 1 (The Cat/Mouse Problem)

Suppose the cat population at month k is c_k and the mouse population at month k is m_k, and let $\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix}$ denote the population vector at month k. Suppose

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad \text{where} \quad A = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix},$$

and an initial population vector $\mathbf{x}_0$ is given. Then the population vector $\mathbf{x}_k$ can be computed from

$$\mathbf{x}_k = A^k \mathbf{x}_0,$$

so we want to compute A^k by diagonalizing the matrix A.

Since the characteristic polynomial of A is $p(t) = t^2 - 2.1t + 1.1 = (t-1)(t-1.1)$, we see that the eigenvalues of A are 1 and 1.1. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and so we form the change-of-basis matrix

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

Then we have

$$A = P\Lambda P^{-1}, \quad \text{where} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix},$$

and so

$$A^k = P\Lambda^k P^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1.1)^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

In particular, if $\mathbf{x}_0 = \begin{bmatrix} c_0 \\ m_0 \end{bmatrix}$ is the original population vector, we have

$$\begin{aligned}
\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1.1)^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} c_0 \\ m_0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1.1)^k \end{bmatrix}\begin{bmatrix} 2c_0 - m_0 \\ -3c_0 + 2m_0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2c_0 - m_0 \\ (1.1)^k(-3c_0 + 2m_0) \end{bmatrix} \\
&= (2c_0 - m_0)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-3c_0 + 2m_0)(1.1)^k\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

We can now see what happens as time passes (see the data in Example 6 on pp. 69–70). If $3c_0 = 2m_0$, the second term drops out and the population vector stays constant. If $3c_0 < 2m_0$, the first term still is constant, and the second term increases exponentially, but note that the contribution to the mouse population is double the contribution to the cat population. And if $3c_0 > 2m_0$, we see that the population vector decreases exponentially, the mouse population being the first to disappear (why?). ▲

The way we computed $\mathbf{x}_k$ above works in general for any diagonalizable matrix A. The column vectors of P are the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, the entries of Λ^k are $\lambda_1^k, \dots, \lambda_n^k$, and so, letting

$$P^{-1}\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

we have

$$(*)\quad A^k\mathbf{x}_0 = P\Lambda^k(P^{-1}\mathbf{x}_0) = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n.$$

This formula will contain all the information we need, and we will see physical interpretations of analogous formulas when we discuss systems of differential equations in Chapter 7.

EXAMPLE 2 (The Fibonacci Sequence)

We first met the Fibonacci sequence,

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \dots,$$

in Example 9 on p. 74. Each term (starting with the third) is obtained by adding the preceding two: If we let a_k denote the k^{th} number in the sequence, then

$$a_{k+2} = a_k + a_{k+1}, \qquad a_0 = a_1 = 1.$$

Thus, if we define $\mathbf{x}_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, $k \geq 0$, then we can encode the pattern of the sequence in the matrix equation

$$\begin{bmatrix} a_{k+1} \\ a_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}, \qquad k \geq 1.$$

In other words, setting

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix},$$

we have

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for all } k \geq 0, \quad \text{with} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Once again, by computing the powers of the matrix A, we can calculate $\mathbf{x}_k = A^k\mathbf{x}_0$, and hence the k^{th} term in the Fibonacci sequence.

The characteristic polynomial of A is $p(t) = t^2 - t - 1$, and so the eigenvalues are

$$\lambda_1 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1+\sqrt{5}}{2}.$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Then

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix},$$

so we have

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_2 - 1 \\ 1-\lambda_1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -\lambda_1 \\ \lambda_2 \end{bmatrix}.$$

Now we use the formula (∗) above to calculate

$$\begin{aligned}\mathbf{x}_k = A^k\mathbf{x}_0 &= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 \\ &= -\frac{\lambda_1}{\sqrt{5}}\lambda_1^k\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \frac{\lambda_2}{\sqrt{5}}\lambda_2^k\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.\end{aligned}$$

In particular, reading off the first coordinate of this vector, we find that the k^{th} number in the Fibonacci sequence is

$$a_k = \frac{1}{\sqrt{5}}\left(\lambda_2^{k+1} - \lambda_1^{k+1}\right) = \frac{1}{\sqrt{5}}\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\tfrac{1-\sqrt{5}}{2}\right)^{k+1}\right).$$

It's not completely obvious that each such number is an integer! We would be remiss if we didn't point out one of the classic facts about the Fibonacci sequence: If we take the ratio of successive terms, we get

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{\sqrt{5}}\left(\lambda_2^{k+2} - \lambda_1^{k+2}\right)}{\frac{1}{\sqrt{5}}\left(\lambda_2^{k+1} - \lambda_1^{k+1}\right)}.$$

Now, $|\lambda_1| \approx 0.618$, so $\lim\limits_{k\to\infty} \lambda_1^k = 0$ and we have

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lambda_2 = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

This is the famed *golden ratio*. ▲

EXAMPLE 3 (The Cribbage Match)

In Example 8 on p. 72 we posed the following problem. Suppose that over the years in which Fred and Barney have played cribbage, they have observed that when Fred wins a game, he has a 60% chance of winning the next game, whereas when Barney wins a game, he has only a 55% chance of winning the next game. What is the long-term ratio of games won and lost by Fred?

Let p_k be the probability that Fred wins the k^{th} game and $q_k = 1 - p_k$ the probability that Fred loses the k^{th} game. Then, as we established earlier,

$$\begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix} = \begin{bmatrix} 0.60 & 0.45 \\ 0.40 & 0.55 \end{bmatrix}\begin{bmatrix} p_k \\ q_k \end{bmatrix},$$

and so, letting $\mathbf{x}_k = \begin{bmatrix} p_k \\ q_k \end{bmatrix}$ be the *probability vector* after k games, we have

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad \text{where} \quad A = \begin{bmatrix} 0.60 & 0.45 \\ 0.40 & 0.55 \end{bmatrix}.$$

What is distinctive about the *transition matrix* A, and what characterizes the linear algebra of *Markov processes*, is the fact that the entries of each column of A are nonnegative and sum to 1. Indeed, it follows from this observation that the matrix $A - I$ is singular and hence that 1 is an eigenvalue. Of course, in this case, we can just calculate the eigenvalues and eigenvectors directly: The characteristic polynomial is $p(t) = t^2 - 1.15t + 0.15 = (t-1)(t-0.15)$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.15$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 9 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{so}$$

$$P = \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 1 \\ -8 & 9 \end{bmatrix}.$$

Using the formula (∗) once again, we have

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 = c_1\begin{bmatrix} 9 \\ 8 \end{bmatrix} + c_2(0.15)^k\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}_0.$$

Now something very interesting happens. As $k \to \infty$, $(0.15)^k \to 0$ and

$$\lim_{k\to\infty} \mathbf{x}_k = c_1\begin{bmatrix} 9 \\ 8 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 9 \\ 8 \end{bmatrix},$$

no matter what the original probability vector $\mathbf{x}_0$ happens to be (why?). Thus, in the long run, no matter what the win/loss ratio is at any finite stage, we expect that Fred will win $9/17$ and lose $8/17$ of the games. We will explore more of the general theory of Markov processes next. ▲

3.1 Markov Processes

The material in this subsection is quite optional. The result of Theorem 3.3 is worth understanding, even if one skips its proof (see Exercise 15 for an easier proof of a somewhat weaker result). But it is Corollary 3.4 that is truly useful in lots of the exercises.

We begin with some definitions.

> ***Definition.*** We say a vector $\mathbf{x} \in \mathbb{R}^n$ is a *probability vector* if all its entries are nonnegative and add up to 1, i.e., $x_i \geq 0$ for all $i = 1, \dots, n$, and $\sum_{i=1}^{n} x_i = 1$. We say a square matrix A is a *stochastic matrix* if each of its column vectors is a probability vector. A stochastic matrix A is *regular* if, for some $r \geq 1$, the entries of A^r are all positive.

We begin by making a simple observation.

Lemma 3.1. *If A is a stochastic matrix, then $(1, 1, \dots, 1)$ is an eigenvector of A^{T} with eigenvalue 1. Consequently, 1 is an eigenvalue of A.*

Proof. Since $\sum_{i=1}^{n} a_{ij} = 1$ for $j = 1, \dots, n$, it follows immediately that

$$A^{\mathsf{T}}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Thus, 1 is an eigenvalue of A^{T}. But then, from

$$\det(A - I) = \det(A - I)^{\mathsf{T}} = \det(A^{\mathsf{T}} - I) = 0,$$

we infer that 1 is an eigenvalue of A as well. □

Proposition 3.2. *Suppose A is a regular stochastic matrix. Then the eigenvalue 1 has geometric multiplicity 1.*

Proof. Since A is regular, there is some integer $r \geq 1$ so that $B = A^r$ has all positive entries. By Exercise 6.1.14, the eigenvalue 1 of the matrix B^{T} has geometric multiplicity 1. By Exercise 6.2.9, the eigenvalue 1 of the matrix B has geometric multiplicity 1. But since $\mathbf{E}_A(1) \subset \mathbf{E}_{A^r}(1)$, it must be the case that $\dim \mathbf{E}_A(1) = 1$, as desired. □

Now we are in a position to prove a powerful result. The proof does introduce some mathematics of a different flavor, as it involves a few inequalities and estimates.

Theorem 3.3. *Let A be a regular stochastic $n \times n$ matrix, and assume $n \geq 2$. Then*

$$\lim_{k\to\infty} A^k = \begin{bmatrix} | & | & & | \\ \mathbf{v} & \mathbf{v} & \cdots & \mathbf{v} \\ | & | & & | \end{bmatrix},$$

where $\mathbf{v}$ is the unique eigenvector with eigenvalue 1 that is a probability vector. Furthermore, every other eigenvalue λ satisfies $|\lambda| < 1$.

Remark. The regularity hypothesis is needed here. For example, the identity matrix is obviously stochastic, but the eigenvalue 1 has rather high multiplicity. On the other hand, a stochastic matrix A may have a certain number of 0 entries and still be regular. For example,

$$A = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}$$

is regular, inasmuch as A^2 has all positive entries.

The impact of this theorem is the following:

Corollary 3.4. *If A is an $n \times n$ regular stochastic matrix and $\mathbf{x}_0$ is* any *probability vector, then $\lim_{k\to\infty} A^k\mathbf{x}_0 = \mathbf{v}$; i.e., the unique eigenvector $\mathbf{v}$ with eigenvalue 1 that is a* probability vector *is the limiting solution of $\mathbf{x}_{k+1} = A\mathbf{x}_k$, no matter what probability vector $\mathbf{x}_0$ is chosen as the initial condition.*

Proof of Theorem 3.3. We first show that for any $i = 1, \dots, n$, the difference between the largest and smallest entries of the i^{th} row of A^k approaches 0 as $k \to \infty$. If we denote by $a_{ij}^{(k)}$ the ij-entry of A^k, then

$$a_{ij}^{(k+1)} = \sum_{q=1}^{n} a_{iq}^{(k)} a_{qj}.$$

Denote by M_k and m_k the largest and smallest entries, respectively, of the i^{th} row of A^k;

say $M_k = a_{ir}^{(k)}$ and $m_k = a_{is}^{(k)}$. Then we have

$$\begin{aligned} a_{ij}^{(k+1)} &= a_{is}^{(k)} a_{sj} + \sum_{q \neq s} a_{iq}^{(k)} a_{qj} \leq m_k a_{sj} + M_k \sum_{q \neq s} a_{qj} \\ &\leq m_k a_{sj} + M_k(1 - a_{sj}), \quad \text{and} \\ a_{i\ell}^{(k+1)} &= a_{ir}^{(k)} a_{rj} + \sum_{q \neq r} a_{iq}^{(k)} a_{qj} \\ &\geq M_k a_{r\ell} + m_k(1 - a_{r\ell}). \end{aligned}$$

Choose j so that $M_{k+1} = a_{ij}^{(k+1)}$ and ℓ so that $m_{k+1} = a_{i\ell}^{(k+1)}$. Then we have

$$(\dagger) \qquad \begin{aligned} M_{k+1} - m_{k+1} &\leq m_k a_{sj} + M_k(1 - a_{sj}) - M_k a_{r\ell} - m_k(1 - a_{r\ell}) \\ &= (M_k - m_k)(1 - a_{r\ell} - a_{sj}). \end{aligned}$$

Assume for a moment that all the entries of A are positive, and denote the smallest entry of A by α. Then we have $\alpha \leq 1/n$ (why?) and so $0 \leq 1 - 2\alpha < 1$. Then, using (†), we have

$$M_{k+1} - m_{k+1} \leq (M_k - m_k)(1 - 2\alpha),$$

and so

$$M_k - m_k \leq (1 - 2\alpha)^{k-1}(M_1 - m_1),$$

which approaches 0 as $k \to \infty$. This tells us that as k gets very large, the elements of the i^{th} row of A^k are very close to one another, say to $a_{i1}^{(k)}$. That is, the column vectors of A^k are all very close to a single vector $\boldsymbol{\xi}^{(k)}$. Now let $\mathbf{x}$ be an eigenvector of A with eigenvalue 1. Then $A^k\mathbf{x} = \mathbf{x}$, but $A^k\mathbf{x}$ is also very close to

$$\begin{bmatrix} | & | & & | \\ \boldsymbol{\xi}^{(k)} & \boldsymbol{\xi}^{(k)} & \cdots & \boldsymbol{\xi}^{(k)} \\ | & | & & | \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \left(\sum_{i=1}^{n} x_i \right) \begin{bmatrix} | \\ \boldsymbol{\xi}^{(k)} \\ | \end{bmatrix}.$$

Since $\mathbf{x} \neq \mathbf{0}$, we conclude immediately that $\sum_{i=1}^{n} x_i \neq 0$, and so by multiplying by the suitable scalar, we may assume our eigenvector $\mathbf{x}$ satisfies $\sum_{i=1}^{n} x_i = 1$. But now, since $\mathbf{x}$ is very close to $\boldsymbol{\xi}^{(k)}$ and the latter vector has nonnegative entries, it follows that $\mathbf{x}$ must have nonnegative entries as well, and so $\mathbf{x}$ is the probability vector $\mathbf{v} \in \mathbf{E}(1)$. Now we conclude that as $k \to \infty$,

$$A^k \to \begin{bmatrix} | & | & & | \\ \mathbf{v} & \mathbf{v} & \cdots & \mathbf{v} \\ | & | & & | \end{bmatrix},$$

as we wished to show.

(A slight modification of the argument is required when A may have some 0 entries. Since A is a regular stochastic matrix, there is an $r \geq 1$ so that all the entries of A^r are positive. We then calculate $A^{k+r} = A^k A^r$ analogously; letting α denote the smallest entry of A^r, we have

$$M_{k+r} - m_{k+r} \leq (1 - 2\alpha)(M_k - m_k),$$

and therefore

$$\lim_{p \to \infty} M_{k+pr} - m_{k+pr} = 0 \quad \text{for any } k.$$

Since it follows from (†) that

$$M_{k+1} - m_{k+1} \le M_k - m_k \quad \text{for all } k,$$

we deduce as before that $\lim_{k\to\infty} M_k - m_k = 0$, and the proof proceeds from there.)

We have seen in the course of this argument that the one-dimensional eigenspace $\mathbf{E}(1)$ is spanned by a probability vector $\mathbf{v}$. Let $\mathbf{x}$ be an eigenvector with eigenvalue λ. Suppose $\sum_{i=1}^{n} x_i \ne 0$. Then we may assume $\sum_{i=1}^{n} x_i = 1$ (why?). Then

$$\mathbf{v} = \lim_{k\to\infty} A^k\mathbf{x} = \lim_{k\to\infty} \lambda^k\mathbf{x};$$

this can happen only if $\mathbf{x} = \mathbf{v}$ and $\lambda = 1$. If $\mathbf{x}$ is not a scalar multiple of $\mathbf{v}$, it must be the case that $\sum_{i=1}^{n} x_i = 0$. In this case,

$$\lim_{k\to\infty} A^k\mathbf{x} = \begin{bmatrix} | & | & & | \\ \mathbf{v} & \mathbf{v} & \cdots & \mathbf{v} \\ | & | & & | \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \left(\sum_{i=1}^{n} x_i\right) \begin{bmatrix} | \\ \mathbf{v} \\ | \end{bmatrix} = \mathbf{0},$$

and so we infer from

$$\mathbf{0} = \lim_{k\to\infty} A^k\mathbf{x} = \lim_{k\to\infty} \lambda^k\mathbf{x}$$

that $\lim_{k\to\infty} \lambda^k = 0$, i.e., $|\lambda| < 1$. □

Exercises 6.3

1. Let $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$. Calculate A^k for all $k \ge 1$.

***2.** Each day 30% of the oxygen in the earth's atmosphere is transformed into carbon dioxide and the remaining 70% is unaffected. Similarly, 40% of the carbon dioxide is transformed into oxygen and 60% remains as is. Find the steady-state ratio of oxygen to carbon dioxide.[5]

3. Each month U-Haul trucks are driven among the cities of Atlanta, St. Louis, and Poughkeepsie. 1/2 of the trucks in Atlanta remain there, while the remaining trucks are split evenly between St. Louis and Poughkeepsie. 1/3 of the trucks in St. Louis stay there, 1/2 go to Atlanta, and the remaining 1/6 venture to Poughkeepsie. And 1/5 of the trucks in Poughkeepsie remain there, 1/5 go to St. Louis, and 3/5 go to Atlanta. Show that the distribution of U-Haul trucks approaches a steady state, and find it.

4. Jane, Dick, and Spot are playing Frisbee. Dick is equally likely to throw to Jane or Spot; Jane always throws the Frisbee to Spot; and Spot is three times as likely to bring the Frisbee to Jane as to Dick. In the long run, what is the probability that Dick gets the Frisbee? (Be sure to check that the transition matrix here is regular.)

[5] The authors hasten to point out that the data appearing in this exercise have no basis in scientific reality.

*5. Suppose each of two tubs contains two bottles of beer; two are Budweiser and two are Beck's. Each minute, Fraternity Freddy picks a bottle of beer from each tub at random and replaces it in the other tub. After a long time, what portion of the time will there be exactly one bottle of Beck's in the first tub? at least one bottle of Beck's?

*6. Gambling Gus has \$200 and plays a game where he must continue playing until he has either lost all his money or doubled it. In each game, he has a $2/5$ chance of winning \$100 and a $3/5$ chance of losing \$100. What is the probability that he eventually loses all his money? (Warning: The stochastic matrix here is far from regular, so there is no steady state. A calculator or computer is required.)

*7. If $a_0 = 2$, $a_1 = 3$, and $a_{k+1} = 3a_k - 2a_{k-1}$, for all $k \geq 1$, use methods of linear algebra to determine the formula for a_k.

8. If $a_0 = a_1 = 1$ and $a_{k+1} = a_k + 6a_{k-1}$ for all $k \geq 1$, use methods of linear algebra to determine the formula for a_k.

9. Suppose $a_0 = 0$, $a_1 = 1$, and $a_{k+1} = 3a_k + 4a_{k-1}$ for all $k \geq 1$. Use methods of linear algebra to find the formula for a_k.

10. If $a_0 = 0$, $a_1 = 1$, and $a_{k+1} = 4a_k - 4a_{k-1}$ for all $k \geq 1$, use methods of linear algebra to determine the formula for a_k. (*Hint:* The matrix will not be diagonalizable, but you can get close if you stare at Exercise 6.2.7.)

*11. If $a_0 = 0$, $a_1 = a_2 = 1$, and $a_{k+1} = 2a_k + a_{k-1} - 2a_{k-2}$ for $k \geq 2$, use methods of linear algebra to determine the formula for a_k.

12. Consider the cat/mouse population problem studied in Example 1. Solve the following versions, including an investigation of the dependence on the original populations.

a. $$\begin{aligned} c_{k+1} &= 0.7c_k + 0.1m_k \\ m_{k+1} &= -0.2c_k + m_k \end{aligned}$$

*b. $$\begin{aligned} c_{k+1} &= 1.3c_k + 0.2m_k \\ m_{k+1} &= -0.1c_k + m_k \end{aligned}$$

c. $$\begin{aligned} c_{k+1} &= 1.1c_k + 0.3m_k \\ m_{k+1} &= 0.1c_k + 0.9m_k \end{aligned}$$

What conclusions do you draw?

13. Show that when $\mathbf{x}$ is a probability vector and A is a stochastic matrix, then $A\mathbf{x}$ is another probability vector.

14. Suppose A is a stochastic matrix and $\mathbf{x}$ is an eigenvector with eigenvalue $\lambda \neq 1$. Show directly (i.e., without reference to the proof of Theorem 3.3) that $(1, 1, \ldots, 1) \cdot \mathbf{x} = 0$.

15. a. Let A be a stochastic matrix with positive entries, let $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{y} = A\mathbf{x}$. Show that

$$|y_1| + |y_2| + \cdots + |y_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

and that equality holds if and only if all the (nonzero) entries of $\mathbf{x}$ have the same sign.

b. Show that if A is a stochastic matrix with positive entries and $\mathbf{x}$ is an eigenvector with eigenvalue 1, then all the entries of $\mathbf{x}$ have the same sign.

c. Prove using part *b* that if A is a stochastic matrix with positive entries, then there is a unique probability vector in $\mathbf{E}(1)$ and hence $\dim \mathbf{E}(1) = 1$.

d. Prove that if λ is an eigenvalue of a stochastic matrix with positive entries, then $|\lambda| \leq 1$.

e. Assume A is a diagonalizable, regular stochastic matrix. Prove Theorem 3.3.

4 The Spectral Theorem

We now turn to the study of a large class of diagonalizable matrices, the symmetric matrices. Recall that a square matrix A is symmetric when $A = A^{\mathsf{T}}$. To begin our exploration, let's start with a general *symmetric* 2×2 matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

whose characteristic polynomial is $p(t) = t^2 - (a+c)t + (ac - b^2)$. By the quadratic formula, its eigenvalues are

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}.$$

The first thing we notice here is that both eigenvalues are *real* (because the expression under the radical is a sum of squares). When A is not diagonal to begin with, $b \neq 0$, and so the eigenvalues of A are necessarily distinct. Thus, in all instances, the symmetric matrix A is diagonalizable. Moreover, the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix};$$

note that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0,$$

and so the eigenvectors are orthogonal. Since there is an orthogonal basis for $\mathbb{R}^2$ consisting of eigenvectors of A, we of course have an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A. That is, by an appropriate rotation of the usual basis, we obtain a diagonalizing basis for A.

EXAMPLE 1

The eigenvalues of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

are $\lambda_1 = 2$ and $\lambda_2 = -3$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

By making these vectors unit vectors, we obtain an orthonormal basis

$$\mathbf{q}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

See Figure 4.1. ▲

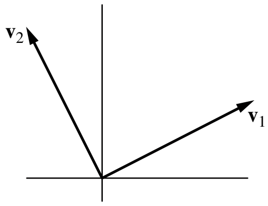

FIGURE 4.1

In general, we have the following important result. Its name comes from the word *spectrum*, which is associated with the physical concept of decomposing light into its component colors.

Theorem 4.1 (Spectral Theorem). *Let A be a symmetric $n \times n$ matrix. Then*

1. *The eigenvalues of A are real.*
2. *There is an orthonormal basis $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of A. That is, there is an orthogonal matrix Q so that $Q^{-1}AQ = \Lambda$ is diagonal.*

Before we get to the proof, we recall a salient feature of symmetric matrices. From Proposition 5.2 of Chapter 2 we recall that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $n \times n$ matrices A we have

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^{\mathsf{T}}\mathbf{y}.$$

In particular, when A is symmetric,

$$\boxed{A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}.}$$

Proof. We begin by proving that the eigenvalues of a symmetric matrix must be real. The proof begins with a trick to turn complex entities into real. Let $\lambda = a + bi$ be a (potentially complex) eigenvalue of A, and consider the *real* matrix

$$\begin{aligned} S = \big(A - (a+bi)I\big)\big(A - (a-bi)I\big) &= A^2 - 2aA + (a^2 + b^2)I \\ &= (A - aI)^2 + b^2 I. \end{aligned}$$

(This is just the usual “multiply by the conjugate” trick from high school algebra.) Since $\det(A - \lambda I) = 0$, it follows that[6] $\det S = 0$. Thus S is singular, and so there is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that $S\mathbf{x} = \mathbf{0}$.

Since $S\mathbf{x} = \mathbf{0}$, the dot product $S\mathbf{x} \cdot \mathbf{x} = 0$. Therefore,

$$\begin{aligned} 0 = S\mathbf{x} \cdot \mathbf{x} &= \Big(\big((A - aI)^2 + b^2 I\big)\mathbf{x}\Big) \cdot \mathbf{x} \\ &= (A - aI)\mathbf{x} \cdot (A - aI)\mathbf{x} + b^2\mathbf{x} \cdot \mathbf{x} \qquad \text{(using symmetry)} \\ &= \|\big(A - aI\big)\mathbf{x}\|^2 + b^2\|\mathbf{x}\|^2. \end{aligned}$$

Now, the only way the sum of two nonnegative numbers can be zero is for both of them to be zero. That is, since $\mathbf{x} \neq \mathbf{0}$, $\|\mathbf{x}\|^2 \neq 0$, and we infer that $b = 0$ and $(A - aI)\mathbf{x} = \mathbf{0}$. So $\lambda = a$ is a real number, and $\mathbf{x}$ is the corresponding (real) eigenvector.

Now we proceed to prove the second part of the theorem. Let λ_1 be one of the eigenvalues of A, and choose a *unit vector* $\mathbf{q}_1$ that is an eigenvector with eigenvalue λ_1. (Obviously,

[6] Here we are using the fact that the product rule for determinants, Theorem 1.5 of Chapter 5, holds for matrices with complex entries. We certainly have not proved this, but all the results in Chapter 5 work just fine for matrices with complex entries. For a different argument, see Exercise 7.1.9.

this is no problem. We pick an eigenvector and then make it a unit vector by dividing by its length.) Choose $\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ to be any orthonormal basis for $\big(\text{Span}\,(\mathbf{q}_1)\big)^\perp$. What, then, is the matrix for the linear transformation μ_A with respect to the new (*orthonormal*) basis $\{\mathbf{q}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$? It looks like

$$B = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right],$$

for some $(n-1) \times (n-1)$ matrix C and some entries $*$. By the change-of-basis formula, we have

$$B = Q^{-1}AQ = Q^{\mathsf{T}}AQ,$$

because Q is an orthogonal matrix. Therefore,

$$B^{\mathsf{T}} = (Q^{\mathsf{T}}AQ)^{\mathsf{T}} = Q^{\mathsf{T}}A^{\mathsf{T}}Q = Q^{\mathsf{T}}AQ = B.$$

Since B is symmetric, we deduce that the entries $*$ are all 0 and that C is likewise symmetric. We now consider the $(n-1) \times (n-1)$ symmetric matrix C: A unit length eigenvector of C in $\mathbb{R}^{n-1}$ corresponds to a unit vector $\mathbf{q}_2$ in the $(n-1)$-dimensional subspace $\big(\text{Span}\,(\mathbf{q}_1)\big)^\perp$. Continuing this process $n-2$ steps further, we arrive at an orthonormal basis $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ consisting of eigenvectors of A. □

EXAMPLE 2

Consider the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Its characteristic polynomial is $p(t) = -t^3 + 2t^2 + t - 2 = -(t^2-1)(t-2) = -(t+1)(t-1)(t-2)$, so the eigenvalues of A are -1, 1, and 2. As the reader can check, the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that these three vectors form an orthogonal basis for $\mathbb{R}^3$, and we can easily obtain an orthonormal basis by making them unit vectors:

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{q}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The orthogonal diagonalizing matrix Q is therefore

$$Q = \begin{bmatrix} -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

▲

EXAMPLE 3

Consider the symmetric matrix

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}.$$

Its characteristic polynomial is $p(t) = -t^3 + 18t^2 - 81t = -t(t-9)^2$, so the eigenvalues of A are 0, 9, and 9. It is easy to check that

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

gives a basis for $\mathbf{E}(0) = \mathbf{N}(A)$. As for $\mathbf{E}(9)$, we find

$$A - 9I = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix},$$

which has rank 1, and so, as the Spectral Theorem guarantees, $\mathbf{E}(9)$ is two-dimensional, with basis

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

If we want an orthogonal (or orthonormal) basis, we must use the Gram-Schmidt process, Theorem 2.4 of Chapter 4: We take $\mathbf{w}_2 = \mathbf{v}_2$ and let

$$\mathbf{w}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{w}_2}\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix}.$$

It is convenient to eschew fractions, and so we let

$$\mathbf{w}_3' = 2\mathbf{w}_3 = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}.$$

As a check, note that $\mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3'$ do in fact form an orthogonal basis. As before, if we want the orthogonal diagonalizing matrix Q, we must make these vectors unit vectors, so we take

$$\mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{q}_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix},$$

whence

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}.$$

We reiterate that repeated eigenvalues cause no problem with symmetric matrices. ▲

We conclude this discussion with a comparison to our study of projections in Chapter 4. Note that if we write out $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathsf{T}}$, we see, reasoning as in Exercise 2.5.4, that

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} — & \mathbf{q}_1^{\mathsf{T}} & — \\ — & \mathbf{q}_2^{\mathsf{T}} & — \\ & \vdots & \\ — & \mathbf{q}_n^{\mathsf{T}} & — \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^{\mathsf{T}}.$$

This is the so-called *spectral decomposition* of A: Multiplying by a symmetric matrix A is the same as taking a weighted sum (weighted by the eigenvalues) of projections onto the respective eigenspaces. (See Proposition 2.3 of Chapter 4.) This is, indeed, a beautiful result with many applications in higher mathematics and physics.

4.1 Conics and Quadric Surfaces: A Brief Respite from Linearity

We now use the Spectral Theorem to analyze the equations of conic sections and quadric surfaces.

EXAMPLE 4

Suppose we are given the quadratic equation

$$x_1^2 + 4x_1x_2 - 2x_2^2 = 6$$

to graph. Notice that we can write the quadratic expression

$$x_1^2 + 4x_1x_2 - 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^{\mathsf{T}} A\mathbf{x},$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

is the symmetric matrix we analyzed in Example 1 above. Thus, we know that

$$A = Q\Lambda Q^{\mathsf{T}}, \quad \text{where} \quad Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

So, if we make the substitution $\mathbf{y} = Q^{\mathsf{T}}\mathbf{x}$, then we have

$$\mathbf{x}^{\mathsf{T}} A\mathbf{x} = \mathbf{x}^{\mathsf{T}}(Q\Lambda Q^{\mathsf{T}})\mathbf{x} = (Q^{\mathsf{T}}\mathbf{x})^{\mathsf{T}}\Lambda(Q^{\mathsf{T}}\mathbf{x}) = \mathbf{y}^{\mathsf{T}}\Lambda\mathbf{y} = 2y_1^2 - 3y_2^2.$$

Note that the conic is much easier to understand in the y_1y_2-coordinates. Indeed, we recognize that the equation $2y_1^2 - 3y_2^2 = 6$ can be written in the form

$$\frac{y_1^2}{3} - \frac{y_2^2}{2} = 1,$$

from which we see that this is a hyperbola with asymptotes $y_2 = \pm\sqrt{\frac{2}{3}}y_1$, as pictured in Figure 4.2. Now recall that the y_1y_2-coordinates are the coordinates with respect to the

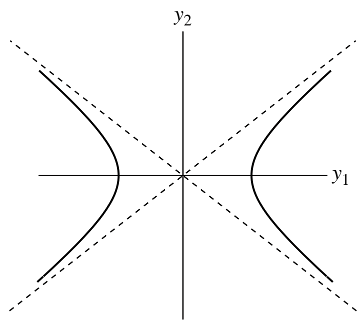

FIGURE 4.2

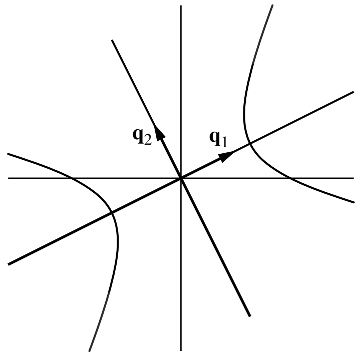

FIGURE 4.3

basis formed by the column vectors of Q. Thus, if we want to sketch the picture in the original x_1x_2-coordinates, we first draw in the basis vectors $\mathbf{q}_1$ and $\mathbf{q}_2$, and these establish the y_1- and y_2-axes, respectively, as shown in Figure 4.3. ▲

We can play this same game with any quadratic equation

$$(\dagger) \qquad \alpha x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2 = \delta,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers. Now we set

$$A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix},$$

so that our equation ($\dagger$) can be written as $\mathbf{x}^\mathsf{T} A\mathbf{x} = \delta$. Since A is symmetric, we can find a diagonal matrix Λ and an orthogonal matrix Q so that $A = Q\Lambda Q^\mathsf{T}$. Thus, setting $\mathbf{y} = Q^\mathsf{T}\mathbf{x}$, we can rewrite equation ($\dagger$) as $\mathbf{y}^\mathsf{T}\Lambda\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 = \delta$.

It's worth recalling that the equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

represents an ellipse (with semiaxes a and b), whereas the equation

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$$

represents a hyperbola with vertices $(\pm a, 0)$ and asymptotes $x_2 = \pm\frac{b}{a}x_1$. We now infer that when our coefficient matrix A has rank 2 (so that both λ_1 and λ_2 are nonzero), our equation ($\dagger$) represents an ellipse or a hyperbola in a rotated coordinate system. (For a continued discussion of conic sections—and of the origin of this nomenclature—we refer the interested reader to Section 2.2 of Chapter 7.)

Now we move on briefly to the three-dimensional setting. Quadric surfaces include those shown in Figure 4.4: ellipsoids, cylinders, and hyperboloids of one and two sheets.

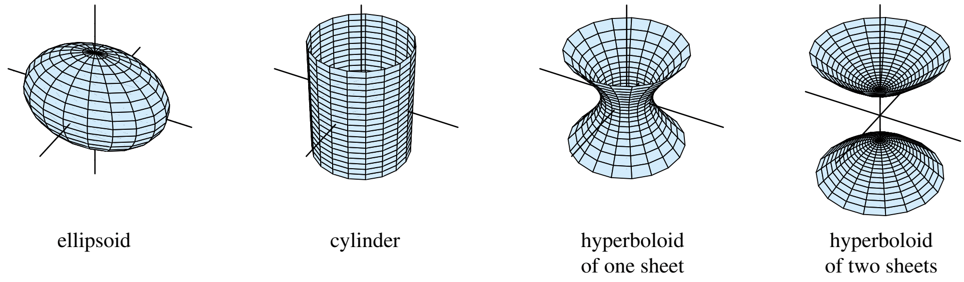

FIGURE 4.4

There are also paraboloids (both elliptic and hyperbolic), but we will address these a bit later. The standard equations to recognize are these:[7]

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1 \qquad \text{ellipsoid}$$

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1 \qquad \text{elliptical cylinder}$$

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}-\frac{x_3^2}{c^2}=1 \qquad \text{hyperboloid of one sheet}$$

$$-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1 \qquad \text{hyperboloid of two sheets}$$

If we begin with any quadratic equation in three variables, we can proceed as we did with two variables, beginning by writing a symmetric coefficient matrix and finding a rotated coordinate system in which we recognize a "standard" equation.

We now turn to another example.

EXAMPLE 5

Consider the surface defined by the equation

$$2x_1x_2 + 2x_1x_3 + x_2^2 + x_3^2 = 2.$$

We observe that if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

is the symmetric matrix from Example 2, then

$$\mathbf{x}^{\mathsf{T}}A\mathbf{x} = 2x_1x_2 + 2x_1x_3 + x_2^2 + x_3^2,$$

and so we use the diagonalization and the substitution $\mathbf{y} = Q^{\mathsf{T}}\mathbf{x}$ as before to write

$$\mathbf{x}^{\mathsf{T}}A\mathbf{x} = \mathbf{y}^{\mathsf{T}}\Lambda\mathbf{y}, \quad \text{where} \quad \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

[7]To remember which hyperboloid equation is which, it helps to solve for x_3^2: In the case of one sheet, we get elliptical cross sections for all values of x_3; in the case of two sheets, we see that $|x_3| \geq |c|$.

that is, in terms of the coordinates $\mathbf{y} = (y_1, y_2, y_3)$, we have

$$2x_1x_2 + 2x_1x_3 + x_2^2 + x_3^2 = -y_1^2 + y_2^2 + 2y_3^2,$$

and the graph of $-y_1^2 + y_2^2 + 2y_3^2 = 2$ is the hyperboloid of one sheet shown in Figure 4.5. This is the picture with respect to the "new basis" $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ (given in the solution of

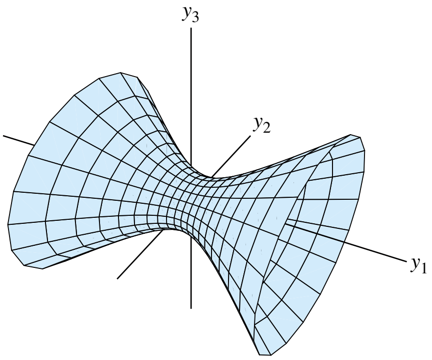

FIGURE 4.5

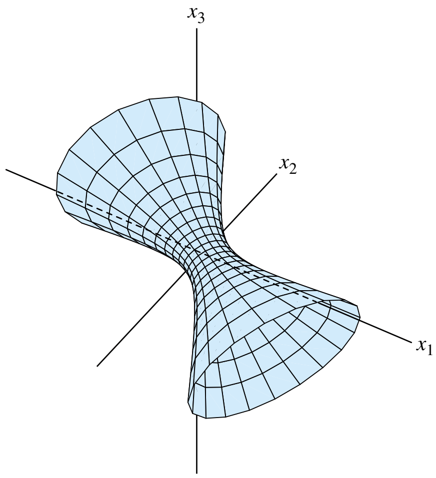

FIGURE 4.6

Example 2). The picture with respect to the standard basis, then, is as shown in Figure 4.6. (This figure is obtained by applying the linear transformation $\mu_Q : \mathbb{R}^3 \to \mathbb{R}^3$. Why?) ▲

The alert reader may have noticed that we're lacking certain curves and surfaces given by quadratic equations. If there are linear terms present along with the quadratic, we must adjust accordingly. For example, we recognize that

$$x_1^2 + 2x_2^2 = 1$$

is the equation of an ellipse centered at the origin. Correspondingly, by completing the square twice, we see that

$$x_1^2 + 2x_1 + 2x_2^2 - 3x_2 = \tfrac{13}{2}$$

is the equation of a congruent ellipse centered at $(-1, \frac{3}{4})$. However, the linear terms become all-important when the symmetric matrix defining the quadratic terms is singular. For example,

$$x_1^2 - x_1 = 1$$

defines a pair of lines, whereas

$$x_1^2 - x_2 = 1$$

defines a parabola. (See Figure 4.7.)

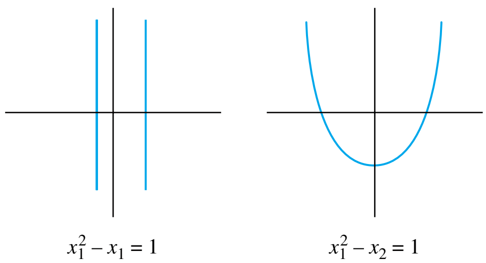

FIGURE 4.7

EXAMPLE 6

We wish to sketch the surface

$$5x_1^2 - 8x_1x_2 - 4x_1x_3 + 5x_2^2 - 4x_2x_3 + 8x_3^2 + 2x_1 + 2x_2 + x_3 = 9.$$

No, we did not pull this mess out of a hat. The quadratic terms came, as might be predicted, from Example 3. Thus, we make the change of coordinates given by $\mathbf{y} = Q^{\mathsf{T}}\mathbf{x}$, with

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}.$$

Since $\mathbf{x} = Q\mathbf{y}$, we have

$$2x_1 + 2x_2 + x_3 = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} Q\mathbf{y} = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1,$$

and so our given equation becomes, in the $y_1y_2y_3$-coordinates,

$$9y_2^2 + 9y_3^2 + 3y_1 = 9.$$

Rewriting this a bit, we have

$$y_1 = 3(1 - y_2^2 - y_3^2),$$

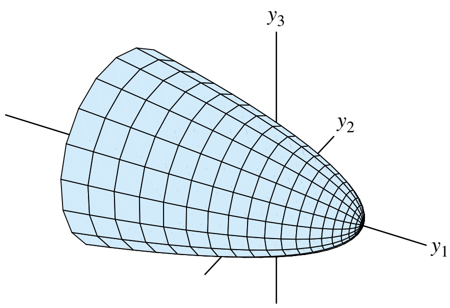

FIGURE 4.8

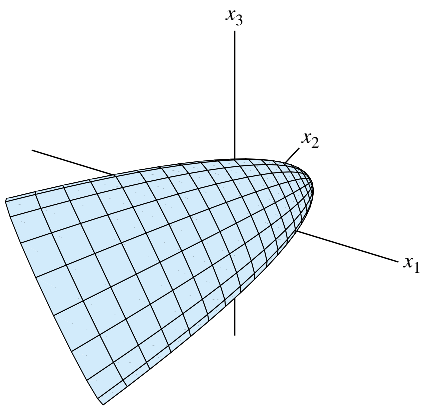

FIGURE 4.9

which we recognize as a (circular) paraboloid, shown in Figure 4.8. The sketch of the surface in our original $x_1x_2x_3$-coordinates is then as shown in Figure 4.9. ▲

Exercises 6.4

1. Find orthogonal matrices that diagonalize each of the following symmetric matrices.

*a. $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

*d. $\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & -1 \\ -2 & -1 & -1 \end{bmatrix}$

*e. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

***2.** Suppose A is a symmetric matrix with eigenvalues 2 and 5. If the vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ span the 5-eigenspace, what is $A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$? Give your reasoning.

3. A symmetric matrix A has eigenvalues 1 and 2. Find A if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ spans $\mathbf{E}(2)$.

4. Suppose A is symmetric, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $\det A = 6$. Give the matrix A. Explain your reasoning clearly. (*Hint:* What are the eigenvalues of A?)

5. Decide (as efficiently as possible) which of the following matrices are diagonalizable. Give your reasoning.

$$A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Let A be a symmetric matrix. Without using the Spectral Theorem, show that if $\lambda \neq \mu$, $\mathbf{x} \in \mathbf{E}(\lambda)$, and $\mathbf{y} \in \mathbf{E}(\mu)$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

***7.** Show that if λ is the only eigenvalue of a symmetric matrix A, then $A = \lambda I$.

8. Suppose A is a diagonalizable matrix whose eigenspaces are orthogonal. Prove that A is symmetric.

9. a. Suppose A is a symmetric $n \times n$ matrix. Using the Spectral Theorem, prove that if $A\mathbf{x} \cdot \mathbf{x} = 0$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then $A = O$.

b. Give an example to show that the hypothesis of symmetry is needed in part *a*.

10. Apply the Spectral Theorem to establish that any symmetric matrix A satisfying $A^2 = A$ is in fact a projection matrix.

11. a. Suppose A is a symmetric $n \times n$ matrix satisfying $A^4 = I$. Use the Spectral Theorem to give a complete description of $\mu_A : \mathbb{R}^n \to \mathbb{R}^n$. (*Hint:* For starters, what are the potential eigenvalues of A?)

b. What happens for a symmetric $n \times n$ matrix satisfying $A^k = I$ for some integer $k \geq 2$?

12. We say a symmetric matrix A is *positive definite* if $A\mathbf{x}\cdot\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, *negative definite* if $A\mathbf{x}\cdot\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$, and *positive* (resp., *negative*) *semidefinite* if $A\mathbf{x}\cdot\mathbf{x} \geq 0$ (resp., ≤ 0) for all $\mathbf{x}$.

a. Show that if A and B are positive (negative) definite, then so is $A + B$.

b. Show that A is positive (resp., negative) definite if and only if all its eigenvalues are positive (resp., negative).

c. Show that A is positive (resp., negative) semidefinite if and only if all its eigenvalues are nonnegative (resp., nonpositive).

d. Show that if C is any $m \times n$ matrix of rank n, then $A = C^{\mathsf{T}}C$ has positive eigenvalues.

e. Prove or give a counterexample: If A and B are positive definite, then so is $AB + BA$.

13. Let A be an $n \times n$ matrix. Show that A is nonsingular if and only if every eigenvalue of $A^{\mathsf{T}}A$ is positive.

14. Prove that if A is a positive semidefinite (symmetric) matrix (see Exercise 12 for the definition), then there is a unique positive semidefinite (symmetric) matrix B with $B^2 = A$.

15. Suppose A and B are symmetric and $AB = BA$. Prove there is an orthogonal matrix Q so that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. (*Hint:* Let λ be an eigenvalue of A. Use the Spectral Theorem to show that there is an orthonormal basis for $\mathbf{E}(\lambda)$ consisting of eigenvectors of B.)

16. Sketch the following conic sections, giving axes of symmetry and asymptotes (if any).

a. $6x_1x_2 - 8x_2^2 = 9$

*b. $3x_1^2 - 2x_1x_2 + 3x_2^2 = 4$

*c. $16x_1^2 + 24x_1x_2 + 9x_2^2 - 3x_1 + 4x_2 = 5$

d. $10x_1^2 + 6x_1x_2 + 2x_2^2 = 11$

e. $7x_1^2 + 12x_1x_2 - 2x_2^2 - 2x_1 + 4x_2 = 6$

17. Sketch the following quadric surfaces.

*a. $3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3 = 4$

b. $4x_1^2 - 2x_1x_2 - 2x_1x_3 + 3x_2^2 + 4x_2x_3 + 3x_3^2 = 6$

c. $-x_1^2 + 2x_2^2 - x_3^2 - 4x_1x_2 - 10x_1x_3 + 4x_2x_3 = 6$

*d. $2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 - x_1 + x_2 + x_3 = 1$

e. $3x_1^2 + 4x_1x_2 + 8x_1x_3 + 4x_2x_3 + 3x_3^2 = 8$

f. $3x_1^2 + 2x_1x_3 - x_2^2 + 3x_3^2 + 2x_2 = 0$

18. Let $a, b, c \in \mathbb{R}$, and let $f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$.

a. The Spectral Theorem tells us that there exists an orthonormal basis for $\mathbb{R}^2$ with respect to whose coordinates (y_1, y_2) we have

$$f(x_1, x_2) = \tilde{f}(y_1, y_2) = \lambda y_1^2 + \mu y_2^2.$$

Show that the y_1y_2-axes are obtained by rotating the x_1x_2-axes through an angle α, where

$$\cot 2\alpha = \frac{a - c}{2b}.$$

Determine the type (ellipse, hyperbola, etc.) of the conic section $f(x_1, x_2) = 1$ from a, b, and c. (*Hint:* Use the characteristic polynomial to eliminate λ^2 in your computation of $\tan 2\alpha$.)

b. Use the formula for $\tilde{f}$ above to find the maximum and minimum of $f(x_1, x_2)$ on the unit circle $x_1^2 + x_2^2 = 1$.

HISTORICAL NOTES

Although we have presented an analysis of quadratic forms as an application of the notions of eigenvalues and eigenvectors, the historical development was actually quite the opposite. Eigenvalues and eigenvectors were first discovered in the context of quadratic forms.

In the late 1700s Joseph-Louis Lagrange (1736–1813) attempted to prove that the solar system was stable—that is, that the planets would not ever widely deviate from their orbits. Lagrange modeled planetary motion using differential equations. He was assisted in his effort by Pierre-Simon Laplace (1749–1827). Together they reduced the solution of the differential equations to what in actuality was an eigenvalue problem for a matrix of coefficients determined by their knowledge of the planetary orbits. Without having any official notion of matrices, they constructed a quadratic form from the array of coefficients and essentially uncovered the eigenvalues and eigenvectors of the matrix by studying the quadratic form. In fact, they made great progress on the problem but were not able to complete a proof of stability. Indeed, this remains an open question!

Earlier work in quadratic forms was led by Gottfried Leibniz (1646–1716) and Leonhard Euler (1707–1783). The work of Carl Freidrich Gauss (1777–1855) in the early nineteenth century brought together many results on quadratic forms, their determinants, and their diagonalization. In the latter half of the 1820s, it was Augustin-Louis Cauchy (1789–1857), also studying planetary motion, who recognized some common threads throughout the work of Euler, Lagrange, and others. He began to consider the importance of the eigenvalues associated with quadratic forms. Cauchy worked with linear combinations of quadratic forms, $sA + tB$, and discovered interesting properties of the form when s and t were chosen so that $\det(sA + tB) = 0$. He dubbed these special values of s and t characteristic values. The terms *characteristic value* and *characteristic vector* are used in many modern texts as synonyms for eigenvalue and eigenvector. The prefix *eigen* comes from a German word that may be translated, for example, as "peculiar" or "appropriate." You may find eigenvectors and eigenvalues peculiar, but the more relevant translation is either "innate" (since an eigenvalue is something an array is born with) or "own" or "self" (since an eigenvector is one that is mapped on top of itself).

Like the notion of orthogonality discussed in Chapter 4, the concept of eigenvalue has meaning when extended beyond matrices to linear maps on abstract vector spaces, such as spaces of functions. The differential equations studied by Joseph Fourier (1768–1830) (see the Historical Note in Chapter 4) modeling heat diffusion lend themselves to eigenvalue-eigenfunction techniques. These eigenfunctions (eigenvectors of the differential operators as linear maps on the vector spaces consisting of functions) generate all the solutions. (See our discussion of normal modes in Section 3 of Chapter 7.) They arise throughout the study of mathematical physics, explaining the tones and overtones of a guitar string or drumhead, as well as the quantum states of atomic physics.

On a completely different note, we also explored Markov processes a bit in this chapter. Andrei Markov (1856–1922) was one several students of Pafnuty Chebyshev (1821–1894), who made great contributions to the field of probability and statistics. Markov studied sequences of experimental outcomes where the future depended only on the present, not on the past. For example, suppose you have been playing a dice game and are now \$100 in the hole. The amount you will owe after the next roll, the future outcome, is a function only of that roll and your current state of being \$100 in debt. The past doesn't matter—it only matters that you currently owe \$100. This is an example of a Markov chain. Markov himself had purely theoretical motivations in mind, however, when studying such chains of events. He was hoping to find simpler proofs for some of the major results of probability theory.

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C H A P T E R 7

FURTHER TOPICS

The three sections of this chapter treat somewhat more advanced topics. They are essentially independent of one another, although the Jordan canonical form of a matrix, introduced in Section 1, makes a subtle appearance a few times in the subsequent sections. There is, nevertheless, a common theme: eigenvalues, eigenvectors, and their applications. In Section 1 we deal with the troublesome cases that arose during our discussion of diagonalizability in Chapter 6. In Section 2, we learn how to encode rigid motions of two- and three-dimensional space by matrices, an important topic in computer graphics. And last, in Section 3, we will see how our work in Section 3 of Chapter 6 naturally generalizes to the study of systems of differential equations.

1 Complex Eigenvalues and Jordan Canonical Form

In this section, we discuss the two issues that caused us trouble in Section 2 of Chapter 6: complex eigenvalues and the situation where geometric multiplicity is less than algebraic.

Recall that the *complex numbers*, denoted $\mathbb{C}$, are defined to be all numbers of the form $a + bi$, $a, b \in \mathbb{R}$, with addition and multiplication defined as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

(The multiplication rule is easy to remember: We just expand using the distributive law and the rule $i^2 = -1$.)

EXAMPLE 1

$$(a + bi)(a - bi) = \big(a^2 - b(-b)\big) + \big(a(-b) + ba\big)i = a^2 + b^2.$$ ▲

We can visualize $\mathbb{C}$ as $\mathbb{R}^2$, using the numbers 1 and i as a basis. That is, the complex number $a + bi$ corresponds geometrically to the vector $(a, b) \in \mathbb{R}^2$. Given a complex number $z = a + bi$, the *real part* of z (denoted $\text{Re}\, z$) is equal to a, and the *imaginary part* of z (denoted $\text{Im}\, z$) is equal to b. Not surprisingly, complex numbers z with $\text{Im}\, z = 0$ are called real numbers; those with $\text{Re}\, z = 0$ are called (purely) imaginary. The reflection of $z = a + bi$ in the real axis is called its *conjugate* $\overline{z}$; thus, $\overline{a + bi} = a - bi$. The *modulus* of the complex number $z = a + bi$ is the length of the vector (a, b), and is usually denoted $|z|$.

EXAMPLE 2

Let $z = a + bi$ be a nonzero complex number. Then its reciprocal is found as follows:

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}.$$

(Note that $z \neq 0$ means that $a^2 + b^2 > 0$.)

Addition of complex numbers is simply addition of vectors in the plane, but multiplication is far more interesting. Introducing polar coordinates in the plane, as shown in Figure 1.1, we now write $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$. This is often called the *polar form* of the complex number z.

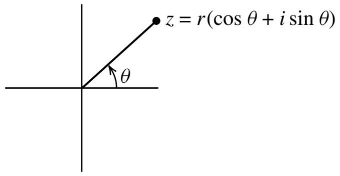

FIGURE 1.1

EXAMPLE 3

Consider the product

$$\big(\sqrt{3}+i\big)\big(2+2\sqrt{3}i\big) = \big((\sqrt{3})(2) - (1)(2\sqrt{3})\big) + \big((\sqrt{3})(2\sqrt{3}) + (1)(2)\big)i = 8i.$$

Now let's look at the picture in Figure 1.2: Using the polar representation, we discover why

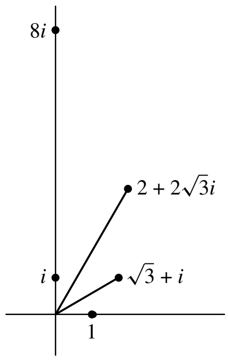

FIGURE 1.2

the product is purely imaginary:

$$\begin{aligned}
\sqrt{3}+i &= 2\big(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\big) = 2\big(\cos(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{6})i\big) \\
2+2\sqrt{3}i &= 4\big(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\big) = 4\big(\cos(\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{3})i\big) \\
(\sqrt{3}+i)(2+2\sqrt{3}i) &= 8\big(\cos(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{6})i\big)\big(\cos(\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{3})i\big) \\
&= 8\Big(\big(\cos(\tfrac{\pi}{6})\cos(\tfrac{\pi}{3}) - \sin(\tfrac{\pi}{6})\sin(\tfrac{\pi}{3})\big) \\
&\qquad + \big(\cos(\tfrac{\pi}{6})\sin(\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{6})\cos(\tfrac{\pi}{6})\big)i\Big) \\
&= 8\big(\cos(\tfrac{\pi}{6}+\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{6}+\tfrac{\pi}{3})i\big) = 8\big(\cos(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{2})i\big) = 8i.
\end{aligned}$$

The experience of the last example lies at the heart of the geometric interpretation of the algebra of complex numbers.

Proposition 1.1. *Let $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$. Then*

$$zw = r\rho\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big).$$

That is, to multiply two complex numbers, we multiply their moduli and add their angles.

Proof. Recall the basic trigonometric formulas

$$\cos(\theta+\phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi \quad \text{and}$$
$$\sin(\theta+\phi) = \sin\theta\,\cos\phi + \cos\theta\,\sin\phi.$$

Now,

$$\begin{aligned} zw &= \big(r(\cos\theta + i\sin\theta)\big)\big(\rho(\cos\phi + i\sin\phi)\big) \\ &= r\rho(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= r\rho\big((\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) + i(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi)\big) \\ &= r\rho\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big), \end{aligned}$$

as required. □

Earlier (see Section 1 of Chapter 1 or Section 6 of Chapter 3) we defined a (real) vector space to be a collection of objects (vectors) that we can add and multiply by (real) scalars, subject to various algebraic rules. We now broaden our definition to allow complex numbers as scalars.

Definition. A *complex vector space* V is a set that is equipped with two operations, vector addition and (complex) scalar multiplication, which satisfy the following properties:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There is $\mathbf{0} \in V$ (the zero vector) so that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.
4. For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u}$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. For all $c, d \in \mathbb{C}$ and $\mathbf{u} \in V$, $c(d\mathbf{u}) = (cd)\mathbf{u}$.
6. For all $c \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
7. For all $c, d \in \mathbb{C}$ and $\mathbf{u} \in V$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
8. For all $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$.

EXAMPLE 4

(a) The most basic example of a complex vector space is the set of all n-tuples of complex numbers, $\mathbb{C}^n = \{(z_1, \dots, z_n) : z_1, \dots, z_n \in \mathbb{C}\}$. Addition and scalar multiplication are defined component by component.

(b) As in Section 6 of Chapter 3, we have the vector space of $m \times n$ matrices with complex entries.

(c) Likewise, we have the vector space of complex-valued continuous functions on the interval $\mathcal{I}$. This vector space plays an important role in differential equations and in the physics and mathematics of waves. ▲

For us, the chief concern here is eigenvalues and eigenvectors. Now that we've expanded our world of scalars to the complex numbers, it is perfectly legitimate for a *complex* scalar λ to be an eigenvalue and for a nonzero vector $\mathbf{v} \in \mathbb{C}^n$ to be an eigenvector of an $n \times n$ (perhaps real) matrix.

EXAMPLE 5

Consider the matrix

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}.$$

We see that the characteristic polynomial of A is $p(t) = t^2 - 2t + 5$, and so, applying the quadratic formula, the eigenvalues are $\frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$. To find the eigenvectors, we follow the usual procedure.

E$(1+2i)$**:** We consider

$$A - (1+2i)I = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}$$

and read off the vector $\mathbf{v}_1 = (1+2i, 1)$ as a basis vector.

E$(1-2i)$**:** Now we consider

$$A - (1-2i)I = \begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix},$$

and the vector $\mathbf{v}_2 = (1-2i, 1)$ gives us a basis.

Note that $\mathbf{v}_2 = \overline{\mathbf{v}}_1$; this should be no surprise, since $A\overline{\mathbf{v}}_1 = \overline{A\mathbf{v}_1} = \overline{(1+2i)\mathbf{v}_1} = (1-2i)\overline{\mathbf{v}}_1$. So, thinking of our matrix as representing a linear map from $\mathbb{C}^2$ to $\mathbb{C}^2$, we see that it can be diagonalized: With respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, the matrix representing μ_A is the diagonal matrix

$$\begin{bmatrix} 1+2i & \\ & 1-2i \end{bmatrix}.$$

We can glean a bit more information about the underlying linear map μ_A from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Then $\mathbf{v}_1 = \mathbf{u}_1 - i\mathbf{u}_2$, and we interpret the eigenvector equation in terms of its real and imaginary parts:

$$A(\mathbf{u}_1 - i\mathbf{u}_2) = A\mathbf{v}_1 = (1+2i)\mathbf{v}_1 = (1+2i)(\mathbf{u}_1 - i\mathbf{u}_2) = (\mathbf{u}_1 + 2\mathbf{u}_2) + i(2\mathbf{u}_1 - \mathbf{u}_2)$$

and so

$$A\mathbf{u}_1 = \mathbf{u}_1 + 2\mathbf{u}_2 \quad \text{and} \quad A\mathbf{u}_2 = -2\mathbf{u}_1 + \mathbf{u}_2.$$

That is, the matrix representing the linear map $\mu_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

▲

EXAMPLE 6

Let's return to the matrix

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{3}+\frac{\sqrt{6}}{6} & \frac{1}{6}-\frac{\sqrt{6}}{3} \\ \frac{1}{3}-\frac{\sqrt{6}}{6} & \frac{2}{3} & \frac{1}{3}+\frac{\sqrt{6}}{6} \\ \frac{1}{6}+\frac{\sqrt{6}}{3} & \frac{1}{3}-\frac{\sqrt{6}}{6} & \frac{1}{6} \end{bmatrix},$$

which we first encountered in Exercise 4.3.24. A "short computation"[1] reveals that the characteristic polynomial of this matrix is $p(t) = -t^3 + t^2 - t + 1 = -(t-1)(t^2+1)$. Thus, A has one real eigenvalue, $\lambda = 1$, and two complex eigenvalues, $\pm i$. We find the complex eigenvectors by considering the linear transformation $\mu_A\colon \mathbb{C}^3 \to \mathbb{C}^3$.

E(i): We find that $\mathbf{v}_1 = \begin{bmatrix} 1-2\sqrt{6}i \\ 2+\sqrt{6}i \\ -5 \end{bmatrix}$ gives a basis for

$$\mathbf{N}(A - iI) = \mathbf{N}\left(\begin{bmatrix} \frac{1}{6}-i & \frac{1}{3}+\frac{\sqrt{6}}{6} & \frac{1}{6}-\frac{\sqrt{6}}{3} \\ \frac{1}{3}-\frac{\sqrt{6}}{6} & \frac{2}{3}-i & \frac{1}{3}+\frac{\sqrt{6}}{6} \\ \frac{1}{6}+\frac{\sqrt{6}}{3} & \frac{1}{3}-\frac{\sqrt{6}}{6} & \frac{1}{6}-i \end{bmatrix}\right).$$

E($-i$): Here $\mathbf{v}_2 = \overline{\mathbf{v}}_1 = \begin{bmatrix} 1+2\sqrt{6}i \\ 2-\sqrt{6}i \\ -5 \end{bmatrix}$ gives a basis for $\mathbf{N}(A + iI)$. We can either calculate this from scratch or reason that $(A + iI)\overline{\mathbf{v}}_1 = \overline{(A - iI)\mathbf{v}_1} = \mathbf{0}$, since A is a matrix with real entries.

E(1): We see that $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ gives a basis for

$$\mathbf{N}(A - I) = \mathbf{N}\left(\begin{bmatrix} -\frac{5}{6} & \frac{1}{3}+\frac{\sqrt{6}}{6} & \frac{1}{6}-\frac{\sqrt{6}}{3} \\ \frac{1}{3}-\frac{\sqrt{6}}{6} & -\frac{1}{3} & \frac{1}{3}+\frac{\sqrt{6}}{6} \\ \frac{1}{6}+\frac{\sqrt{6}}{3} & \frac{1}{3}-\frac{\sqrt{6}}{6} & -\frac{5}{6} \end{bmatrix}\right).$$

We should expect, following the reasoning of Section 2 of Chapter 6, that since A has three distinct complex eigenvalues, we can diagonalize the matrix A working over $\mathbb{C}$. Indeed,

[1]Although it is amusing to do the computations in this example by hand, this might be a reasonable place to give in and use a computer program such as Maple, Mathematica, or MATLAB.

we leave it to the reader to check that, taking

$$P = \begin{bmatrix} 1-2\sqrt{6}i & 1+2\sqrt{6}i & 1 \\ 2+\sqrt{6}i & 2-\sqrt{6}i & 2 \\ -5 & -5 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} i & & \\ & -i & \\ & & 1 \end{bmatrix},$$

we have $A = P\Lambda P^{-1}$.

But does this give us any insight into μ_A as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$? Letting

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 2\sqrt{6} \\ -\sqrt{6} \\ 0 \end{bmatrix},$$

we see that $\mathbf{v}_1 = \mathbf{u}_1 - i\mathbf{u}_2$ and $\mathbf{v}_2 = \mathbf{u}_1 + i\mathbf{u}_2$. Since $A\mathbf{v}_1 = i\mathbf{v}_1$, it now follows that

$$A(\mathbf{u}_1 - i\mathbf{u}_2) = i(\mathbf{u}_1 - i\mathbf{u}_2) = \mathbf{u}_2 + i\mathbf{u}_1,$$

and so, using the fact that $\mathbf{u}_1$, $\mathbf{u}_2$, $A\mathbf{u}_1$, and $A\mathbf{u}_2$ are all real vectors, it must be the case that

$$A\mathbf{u}_1 = \mathbf{u}_2 \quad \text{and} \quad A\mathbf{u}_2 = -\mathbf{u}_1.$$

Furthermore, we notice that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{v}_3$ give an orthogonal basis for $\mathbb{R}^3$. Since $A\mathbf{v}_3 = \mathbf{v}_3$ and the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ have the same length, we now infer that μ_A gives a rotation of $\pi/2$ about the axis spanned by $\mathbf{v}_3$, fixing $\mathbf{v}_3$ and rotating $\mathbf{u}_1$ to $\mathbf{u}_2$ and $\mathbf{u}_2$ to $-\mathbf{u}_1$. ▲

Theorem 2.1 of Chapter 6 is still valid when our scalars are complex numbers, and so Corollary 2.2 of Chapter 6 can be rephrased in our new setting:

Proposition 1.2. *Suppose V is an n-dimensional complex vector space and $T: V \to V$ has n distinct eigenvalues. Then T is diagonalizable.*

It is important to remember that this result holds in the province of *complex* vector spaces, with complex eigenvalues and complex eigenvectors. However, when we start with a linear transformation on a *real* vector space, it is not too hard to extend the reasoning in Example 6 to deduce the following result.

Corollary 1.3. *Suppose A is an $n \times n$* real *matrix with n distinct (possibly complex) eigenvalues. Then A is similar to a "block diagonal" matrix of the form*

$$\begin{bmatrix} \boxed{\begin{matrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{matrix}} & & & & & \\ & \ddots & & & & \\ & & \boxed{\begin{matrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{matrix}} & & & \\ & & & \lambda_{2k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix},$$

where $\alpha_1 \pm \beta_1 i, \dots, \alpha_k \pm \beta_k i$ are the $2k$ complex eigenvalues (with $\beta_j \neq 0$) and $\lambda_{2k+1}, \dots, \lambda_n$ are the real eigenvalues.

Proof. Left to the reader in Exercise 3. □

EXAMPLE 7

We wish to find the "block diagonal" form of the matrix

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 8 & 7 & -12 \\ 6 & 4 & -7 \end{bmatrix}$$

as guaranteed us by Corollary 1.3. The characteristic polynomial of A is $p(t) = -t^3 + 5t^2 - 11t + 15$. Checking for rational roots, we find that $\lambda = 3$ is a real eigenvalue, and so we find that $p(t) = -(t-3)(t^2 - 2t + 5)$; thus, $\lambda = 1 \pm 2i$ are the complex eigenvalues of A. The eigenvectors corresponding to the eigenvalues $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, and $\lambda_3 = 3$, respectively, are

$$\mathbf{v}_1 = \begin{bmatrix} 2+i \\ 7+i \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2-i \\ 7-i \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now, taking

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

we see that $A\mathbf{v}_1 = A(\mathbf{u}_1 - i\mathbf{u}_2) = (1+2i)(\mathbf{u}_1 - i\mathbf{u}_2) = (\mathbf{u}_1 + 2\mathbf{u}_2) + i(2\mathbf{u}_1 - \mathbf{u}_2)$, and so

$$\begin{aligned} A\mathbf{u}_1 &= \mathbf{u}_1 + 2\mathbf{u}_2, \\ A\mathbf{u}_2 &= -2\mathbf{u}_1 + \mathbf{u}_2, \\ A\mathbf{u}_3 &= 3\mathbf{u}_3. \end{aligned}$$

Thus, with respect to the basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, the linear transformation $\mu_A : \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Of course, since $\mathcal{B}$ is not an orthogonal basis (as we had in Example 6), the geometric interpretation of μ_A is a bit more subtle: The line spanned by $\mathbf{u}_3$ is stretched by a factor of 3, and the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ is preserved. ▲

Now we come to the much more subtle issue of what to do when the geometric multiplicity of one eigenvalue (or more) is less than its algebraic multiplicity. To motivate the general arguments, we consider the following example.

EXAMPLE 8

Let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

Then the characteristic polynomial of A is $p(t) = t^2 - 4t + 4 = (t-2)^2$. However, since

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix},$$

we know that $\mathbf{E}(2)$ is only one-dimensional, with basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and so A is not diagonalizable. However, we are fortunate enough to observe that the vector $\mathbf{v}_1$ is obviously in the column space of the matrix $A - 2I$. Therefore, the equation

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has a solution, for example:

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$, we have $A\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2$, and the matrix representing μ_A with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

This is called the *Jordan canonical form* of A.

This argument applies generally to any 2×2 matrix A having eigenvalue λ with algebraic multiplicity 2 and geometric multiplicity 1. Let's show that in this case we *must* have $\mathbf{N}(A - \lambda I) = \mathbf{C}(A - \lambda I)$. Assume not; then we choose a basis $\{\mathbf{v}_1\}$ for $\mathbf{N}(A - \lambda I)$ and a basis $\{\mathbf{v}_2\}$ for $\mathbf{C}(A - \lambda I)$; since $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonparallel, we obtain a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$. What's more, we can write $\mathbf{v}_2 = (A - \lambda I)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$, and so

$$A\mathbf{v}_2 = A(A - \lambda I)\mathbf{x} = (A - \lambda I)(A\mathbf{x})$$

is an element of $\mathbf{C}(A - \lambda I)$. Since $\{\mathbf{v}_2\}$ is a basis for this subspace, we infer that $A\mathbf{v}_2 = c\mathbf{v}_2$ for some scalar c. Thus, the matrix representing μ_A with respect to the basis $\mathcal{B}$ is

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & c \end{bmatrix},$$

contradicting the fact that A is not diagonalizable. Thus, we conclude that $\mathbf{C}(A - \lambda I) = \mathbf{N}(A - \lambda I)$ and proceed as above to conclude that the Jordan canonical form of A is

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

▲

A $k \times k$ matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

(with all its other entries 0) is called a k-dimensional *Jordan block* with eigenvalue λ.

Before proceeding to the general result, we need a fact upon which we stumbled in Example 8.

Lemma 1.4. *Let A be an $n \times n$ matrix, and let λ be any scalar. Then the subspace $V = \mathbf{C}(A - \lambda I)$ has the property that*

$$\textit{whenever } \mathbf{v} \in V, \textit{ it is the case that } A\mathbf{v} \in V.$$

That is, the subspace V is invariant *under μ_A.*

Proof. The one-line proof is left to the reader in Exercise 4. □

Theorem 1.5 (Jordan Canonical Form). *Suppose the characteristic polynomial of an $n \times n$ complex matrix A is $p(t) = \pm(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}$. Then there is a basis $\mathcal{B}$ for $\mathbb{C}^n$ with respect to which the matrix representing μ_A is "block diagonal":*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}.$$

For each $j = 1, \ldots, k$, the sum of the sizes of the Jordan blocks with eigenvalue λ_j is the algebraic multiplicity m_j and the number of Jordan blocks with eigenvalue λ_j is the geometric multiplicity d_j.

Examples and Sketch of Proof. Although we shall not give a complete proof of this result here, we begin by examining what happens when the characteristic polynomial is $p(t) = \pm(t - \lambda)^m$ with $m = 2$ or 3, and then indicate the general argument.[2]

Suppose A is a 2×2 matrix with characteristic polynomial $p(t) = (t - \lambda)^2$. Then λ is an eigenvalue of A (with algebraic multiplicity 2) and $\dim \mathbf{N}(A - \lambda I) \geq 1$. If $\dim \mathbf{N}(A - \lambda I) = 2$, then A is diagonalizable, and we have two 1×1 Jordan blocks in J.

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[2]We learned of this proof, which Strang credits to Filippov, in the appendix to Strang's *Linear Algebra and Its Applications*. It also appears, in far greater detail, in Friedberg, Insel, and Spence. We hope we've made the important ideas clear here.

If $\dim \mathbf{N}(A-\lambda I)=1$, then $\dim \mathbf{C}(A-\lambda I)=1$ as well. As we saw in Example 8, we must have $\mathbf{N}(A-\lambda I)\subset \mathbf{C}(A-\lambda I)$ (or else A would be diagonalizable). Let $\{\mathbf{v}_1\}$ be a basis for $\mathbf{N}(A-\lambda I)$. Then there is a vector $\mathbf{v}_2$ so that $(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1$, and the matrix representing μ_A with respect to the basis $\{\mathbf{v}_1,\mathbf{v}_2\}$ for $\mathbb{C}^2$ is

$$\begin{bmatrix}\lambda & 1\\ & \lambda\end{bmatrix},$$

as required. (The reader should check that $\{\mathbf{v}_1,\mathbf{v}_2\}$ forms a linearly independent set.)

Now suppose A is a 3×3 matrix with characteristic polynomial $p(t)=-(t-\lambda)^3$. If $\dim \mathbf{N}(A-\lambda I)=3$, then A is diagonalizable, and there are three 1×1 Jordan blocks in J. Suppose $\dim \mathbf{N}(A-\lambda I)=2$. Then $\dim \mathbf{C}(A-\lambda I)=1$. Could it happen that $\mathbf{C}(A-\lambda I)\cap \mathbf{N}(A-\lambda I)=\{\mathbf{0}\}$? If so, taking a basis $\{\mathbf{v}_1,\mathbf{v}_2\}$ for $\mathbf{N}(A-\lambda I)$ and $\{\mathbf{v}_3\}$ for $\mathbf{C}(A-\lambda I)$, we know $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. Moreover, by Lemma 1.4, we know that $A\mathbf{v}_3=c\mathbf{v}_3$ for some scalar c. But we already know that $A\mathbf{v}_1=\lambda\mathbf{v}_1$ and $A\mathbf{v}_2=\lambda\mathbf{v}_2$. These results contradict the fact that A is not diagonalizable. Thus, $\mathbf{C}(A-\lambda I)\subset \mathbf{N}(A-\lambda I)$. If we choose $\mathbf{v}_2$ spanning $\mathbf{C}(A-\lambda I)$ and $\{\mathbf{v}_1,\mathbf{v}_2\}$ to be a basis for $\mathbf{N}(A-\lambda I)$, then we know there is a vector $\mathbf{v}_3$ so that $(A-\lambda I)\mathbf{v}_3=\mathbf{v}_2$. We leave it to the reader to check in Exercise 5 that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is linearly independent and that the matrix representing μ_A with respect to this basis is

$$J=\left[\begin{array}{c|cc}\lambda & & \\ \hline & \lambda & 1\\ & & \lambda\end{array}\right].$$

In particular, J contains one 1×1 Jordan block and one 2×2 block.

Last, suppose $\dim \mathbf{N}(A-\lambda I)=1$. We leave it to the reader to prove in Exercise 6 that $\mathbf{N}(A-\lambda I)\subset \mathbf{C}(A-\lambda I)$. Now we apply Lemma 1.4: Thinking of μ_A as a linear transformation from the two-dimensional subspace $\mathbf{C}(A-\lambda I)$ to itself, the proof of Proposition 2.3 of Chapter 6 shows that the characteristic polynomial must be $(t-\lambda)^2$, and so we know there is a basis $\{\mathbf{v}_1,\mathbf{v}_2\}$ for $\mathbf{C}(A-\lambda I)$ with the property that $(A-\lambda I)\mathbf{v}_1=\mathbf{0}$ and $(A-\lambda I)\mathbf{v}_2=\mathbf{v}_1$. Now, since $\mathbf{v}_2\in \mathbf{C}(A-\lambda I)$, there is a vector $\mathbf{v}_3\in\mathbb{C}^3$ so that $(A-\lambda I)\mathbf{v}_3=\mathbf{v}_2$. We claim that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is linearly independent. For suppose that

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}.$$

Multiplying the equation by $A-\lambda I$ twice in succession, we obtain

$$c_2\mathbf{v}_1+c_3\mathbf{v}_2=\mathbf{0}\quad\text{and}\quad c_3\mathbf{v}_1=\mathbf{0},$$

from which we deduce that $c_3=0$, hence $c_2=0$, and hence $c_1=0$, as required. With respect to the basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ for $\mathbb{C}^3$, the matrix representing μ_A becomes

$$J=\begin{bmatrix}\lambda & 1 & \\ & \lambda & 1\\ & & \lambda\end{bmatrix},$$

i.e., one 3×3 Jordan block.

The proof of the general result proceeds by mathematical induction.[3] Of course, the theorem holds when $n = 1$. Now suppose we assume it holds for all $j \times j$ matrices for $j < n$; we must prove it holds for an arbitrary $n \times n$ matrix A. Choose an eigenvalue, λ, of A. Then, by definition, $d = \dim \mathbf{N}(A - \lambda I) \geq 1$, and so $\dim \mathbf{C}(A - \lambda I) = n - d \leq n - 1$. By Lemma 1.4, $\mathbf{C}(A - \lambda I)$ is an invariant subspace and so, by the induction hypothesis, there is a Jordan canonical form J for the restriction of μ_A to this subspace; that is, there is a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-d}\}$ for $\mathbf{C}(A - \lambda I)$ so that the matrix for μ_A with respect to this basis consists of various Jordan blocks. How many blocks are there with eigenvalue λ? If $\dim \big(\mathbf{N}(A - \lambda I) \cap \mathbf{C}(A - \lambda I)\big) = \ell$, then there will be precisely ℓ such blocks, since this is the geometric multiplicity of λ for the restriction of μ_A to the invariant subspace $\mathbf{C}(A - \lambda I)$.

We need to see how to choose d additional vectors so as to obtain a basis for $\mathbb{C}^n$. These will come from two sources:

(i) $d - \ell$ additional eigenvectors; and

(ii) ℓ vectors coming from enlarging (by one row and one column) each of the blocks of J with eigenvalue λ.

The first is easy. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ form a basis for $\mathbf{N}(A - \lambda I) \cap \mathbf{C}(A - \lambda I)$; choose $d - \ell$ further vectors $\mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_d$ so that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_d\}$ is a basis for $\mathbf{N}(A - \lambda I)$. The vectors $\mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_d$ fulfill the first need.

The second is, unfortunately, notationally more complicated. Let's enumerate the basis vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-d}\}$ more carefully:

$$\underbrace{\mathbf{w}_1, \ldots, \mathbf{w}_{j_1}}_{\text{first } \lambda\text{-block}}, \quad \underbrace{\mathbf{w}_{j_1+1}, \ldots, \mathbf{w}_{j_2}}_{\text{second } \lambda\text{-block}}, \quad \ldots, \quad \underbrace{\mathbf{w}_{j_{\ell-1}+1}, \ldots, \mathbf{w}_{j_\ell}}_{\ell^{\text{th}}\ \lambda\text{-block}}, \quad \underbrace{\mathbf{w}_{j_\ell+1}, \ldots, \mathbf{w}_{n-d}}_{\text{remaining blocks of } J}.$$

So $\mathbf{w}_1, \mathbf{w}_{j_1+1}, \ldots, \mathbf{w}_{j_{\ell-1}+1}$ are the ℓ eigenvectors with eigenvalue λ in $\mathbf{C}(A - \lambda I)$, and $\mathbf{w}_{j_1}$, $\mathbf{w}_{j_2}, \ldots, \mathbf{w}_{j_\ell}$ are the "final vectors" in each of the respective blocks. Since each of the latter vectors belongs to $\mathbf{C}(A - \lambda I)$, we can find vectors $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ so that $(A - \lambda I)\mathbf{u}_s = \mathbf{w}_{j_s}$ for $s = 1, \ldots, \ell$. Then in our final matrix representation for μ_A we will still have ℓ Jordan blocks with eigenvalue λ, each of one size larger than appeared in J. We list the $n - d + \ell = n - (d - \ell)$ vectors appropriately, and append the $d - \ell$ eigenvectors $\mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_d$:

$$\underbrace{\mathbf{w}_1, \ldots, \mathbf{w}_{j_1}, \mathbf{u}_1}_{\text{first } \lambda\text{-block}}, \quad \underbrace{\mathbf{w}_{j_1+1}, \ldots, \mathbf{w}_{j_2}, \mathbf{u}_2}_{\text{second } \lambda\text{-block}}, \quad \ldots, \quad \underbrace{\mathbf{w}_{j_{\ell-1}+1}, \ldots, \mathbf{w}_{j_\ell}, \mathbf{u}_\ell}_{\ell^{\text{th}}\ \lambda\text{-block}},$$
$$\underbrace{\mathbf{v}_{\ell+1}, \ldots, \mathbf{v}_d}_{\text{extra eigenvectors}}, \quad \underbrace{\mathbf{w}_{j_\ell+1}, \ldots, \mathbf{w}_{n-d}}_{\text{remaining blocks}}.$$

Once we check that this collection of n vectors is linearly independent, we will have a basis for $\mathbb{C}^n$ with respect to which the matrix for μ_A will indeed be in Jordan canonical form. Since the ideas are not difficult but the notation gets cumbersome, we'll leave this to the reader in Exercise 17. □

Remark. It may be useful to make the following observation. For any eigenvalue μ of the matrix A, if the geometric multiplicity of the eigenvalue μ is d, then $\dim \mathbf{C}(A - \mu I) = n - d$. Suppose $\dim \big(\mathbf{C}(A - \mu I) \cap \mathbf{E}(\mu)\big) = \ell$. Then there will be d Jordan blocks with eigenvalue μ, of which $d - \ell$ are 1×1 and ℓ are larger. The sum of the sizes of all the blocks is, of course, the algebraic multiplicity of the eigenvalue μ.

[3] Actually, we need the formulation called *complete induction*, which allows us to assume that the statement $P(j)$ is valid for *all* positive integers $j \leq k$ in order to deduce the validity of $P(k+1)$. Of course, we must first verify that the statement $P(1)$ is valid.

EXAMPLE 9

Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & -2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

After a bit of work—expanding $\det(A - tI)$ first in cofactors along the first column and then along the first row of the only 4×4 matrix that appears—we find that the characteristic polynomial of A is

$$p(t) = -(t-1)^3(t+1)^2.$$

Thus, the eigenvalues of A are 1 (with algebraic multiplicity 3) and -1 (with algebraic multiplicity 2). Performing row reduction, we determine that

$$A - I = \begin{bmatrix} 0 & 2 & 2 & 1 & -2 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$A + I = \begin{bmatrix} 2 & 2 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this information it is easy to read off bases for $\mathbf{E}(1)$, $\mathbf{C}(A - I)$, $\mathbf{E}(-1)$, and $\mathbf{C}(A + I)$, as follows:

$$\mathbf{E}(1)\text{:} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{C}(A - I)\text{:} \quad \left\{ \begin{bmatrix} 2 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{E}(-1)\mathbf{:}\quad \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\mathbf{C}(A+I)\mathbf{:}\quad \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

In particular, we observe that $\mathbf{E}(1)$ is two-dimensional and $\mathbf{E}(-1)$ is one-dimensional. This is enough to tell us that there will be a 2×2 Jordan block with eigenvalue 1, a 1×1 Jordan block with eigenvalue 1, and a 2×2 Jordan block with eigenvalue -1. Explicitly: $\mathbf{E}(1) \cap \mathbf{C}(A - I)$ is one-dimensional, spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

and, by inspection, the vector

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

satisfies $(A - I)\mathbf{v}_2 = \mathbf{v}_1$. If we take

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

then $\{\mathbf{v}_1, \mathbf{v}_3\}$ gives a basis for $\mathbf{E}(1)$. Then we take

$$\mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

which spans $\mathbf{E}(-1)$, and observe (as we saw in Example 8 and the proof of Theorem 1.5) that $\mathbf{E}(-1) \subset \mathbf{C}(A+I)$, and so we find

$$\mathbf{v}_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

with the property that $(A+I)\mathbf{v}_5 = \mathbf{v}_4$. The theory tells us that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ must be a linearly independent set, hence a basis for $\mathbb{C}^5$, and with respect to this basis we obtain the Jordan canonical form

$$J = \left[\begin{array}{cc|c|cc} 1 & 1 & & & \\ 0 & 1 & & & \\ \hline & & 1 & & \\ \hline & & & -1 & 1 \\ & & & 0 & -1 \end{array}\right].$$

▲

Exercises 7.1

1. Suppose A is a real 2×2 matrix with complex eigenvalues $\alpha \pm \beta i$, and suppose $\mathbf{v} = \mathbf{x} - i\mathbf{y}$ is the eigenvector corresponding to $\alpha + \beta i$. (Here $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.)

a. First, explain why the eigenvalues of A must be complex conjugates.

b. Show that the matrix for μ_A with respect to the basis $\{\mathbf{x}, \mathbf{y}\}$ is

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

2. Find the eigenvalues and eigenvectors of the following real matrices, and give bases with respect to which the matrix is (i) diagonalized as a complex linear transformation; (ii) in the "block diagonal" form provided by Corollary 1.3.

a. $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

*b. $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

*c. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 \\ -1 & -1 & 3 & 3 \\ -1 & 3 & -1 & 3 \end{bmatrix}$

e. $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & 0 & -2 & 3 \end{bmatrix}$

3. Prove Corollary 1.3. (*Hint:* Generalize the argument in Exercise 1.)

4. Prove Lemma 1.4.

*5. Verify that, in the case of a 3×3 matrix A with $\dim \mathbf{N}(A - \lambda I) = 2$ in the proof of Theorem 1.5, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a linearly independent set and that the Jordan canonical form is as given.

6. Prove that if $p(t) = -(t - \lambda)^3$ and $\dim \mathbf{N}(A - \lambda I) = 1$, then we must have $\mathbf{N}(A - \lambda I) \subset \mathbf{C}(A - \lambda I)$. (*Hint:* If $\mathbf{N}(A - \lambda I) \cap \mathbf{C}(A - \lambda I) = \{\mathbf{0}\}$, use the two-dimensional case already proved to deduce that $\dim \mathbf{N}(A - \lambda I) \geq 2$.)

7. Mimic the discussion of the examples in the proof of Theorem 1.5 to analyze the case of a 4×4 matrix A with characteristic polynomial:
   *a. $p(t) = (t - \lambda)^2(t - \mu)^2, \quad (\lambda \neq \mu)$
   b. $p(t) = (t - \lambda)^3(t - \mu), \quad (\lambda \neq \mu)$
   c. $p(t) = (t - \lambda)^4$

8. Determine the Jordan canonical form J of each of the following matrices A; give as well a matrix P so that $J = P^{-1}AP$.

   *a. $A = \begin{bmatrix} 3 & 2 & -3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

   b. $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

   c. $A = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & -1 & -1 & 1 \end{bmatrix}$

   *d. $A = \begin{bmatrix} 0 & -1 & -1 & 2 \\ -3 & -1 & -1 & 3 \\ 1 & 0 & 2 & -1 \\ -2 & -2 & -1 & 4 \end{bmatrix}$

   e. $A = \begin{bmatrix} 2 & 2 & -1 & -2 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

9. Suppose A is an $n \times n$ matrix with all *real* entries and suppose λ is a complex eigenvalue of A, with corresponding *complex* eigenvector $\mathbf{v} \in \mathbb{C}^n$. Set $S = (A - \bar{\lambda} I)(A - \lambda I)$. Prove that $\mathbf{N}(S) \neq \{\mathbf{0}\}$. (*Hint:* Write $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.)

10. If $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, define their *(Hermitian) dot product* by

$$\mathbf{w} \cdot \mathbf{z} = \sum_{j=1}^{n} w_j \bar{z}_j.$$

   a. Check that the following properties hold:
      (i) $\mathbf{w} \cdot \mathbf{z} = \overline{\mathbf{z} \cdot \mathbf{w}}$ for all $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$.
      (ii) $(c\mathbf{w}) \cdot \mathbf{z} = c(\mathbf{w} \cdot \mathbf{z})$ for all $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$ and scalars c.
      (iii) $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{z} = (\mathbf{v} \cdot \mathbf{z}) + (\mathbf{w} \cdot \mathbf{z})$ for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{C}^n$.
      (iv) $\mathbf{z} \cdot \mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathbb{C}^n$ and $\mathbf{z} \cdot \mathbf{z} = 0$ only if $\mathbf{z} = \mathbf{0}$.

   b. Defining the length of a vector $\mathbf{z} \in \mathbb{C}^n$ by $\|\mathbf{z}\| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$, prove the *triangle inequality* for vectors in $\mathbb{C}^n$:

$$\|\mathbf{w} + \mathbf{z}\| \leq \|\mathbf{w}\| + \|\mathbf{z}\| \quad \text{for all } \mathbf{w}, \mathbf{z} \in \mathbb{C}^n.$$

11. (Gerschgorin's Circle Theorem) Let A be a complex $n \times n$ matrix. If λ is an eigenvalue of A, show that λ lies in at least one of the disks $|z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$, $i = 1, \ldots, n$, in $\mathbb{C}$. (*Hint:* Use the triangle inequality from Exercise 10.)

12. Use Exercise 11 to show that any eigenvalue λ of an $n \times n$ complex matrix A is at most the largest sum $\sum_{j=1}^{n} |a_{ij}|$ as i varies from 1 to n and, similarly, at most the largest sum $\sum_{i=1}^{n} |a_{ij}|$ as j varies from 1 to n.

13. Use Exercise 12 to show that any eigenvalue λ of a stochastic matrix (see Section 3.1 of Chapter 6) satisfies $|\lambda| \le 1$.

14. Let $T: \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. We say $\mathbf{v} \in \mathbb{C}^n$ is a *generalized eigenvector* of T with corresponding eigenvalue λ if $\mathbf{v} \neq \mathbf{0}$ and $(T - \lambda I)^k(\mathbf{v}) = \mathbf{0}$ for some positive integer k. Define the generalized λ-eigenspace

$$\tilde{\mathbf{E}}(\lambda) = \{\mathbf{v} \in \mathbb{C}^n : \mathbf{v} \in \mathbf{N}\big((T - \lambda I)^k\big) \text{ for some positive integer } k\}.$$

a. Prove that $\tilde{\mathbf{E}}(\lambda)$ is a subspace of $\mathbb{C}^n$.

b. Prove that $T(\tilde{\mathbf{E}}(\lambda)) \subset \tilde{\mathbf{E}}(\lambda)$.

15. a. Suppose $T(\mathbf{w}) = \lambda\mathbf{w}$. Prove that $(T - \mu I)^k(\mathbf{w}) = (\lambda - \mu)^k\mathbf{w}$.

b. Suppose $\lambda_1, \dots, \lambda_k$ are distinct scalars and $\mathbf{v}_1, \dots, \mathbf{v}_k$ are generalized eigenvectors of T with corresponding eigenvalues $\lambda_1, \dots, \lambda_k$, respectively. (See Exercise 14 for the definition.) Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set. (*Hint:* Let ρ_i be the smallest positive integer so that $(T - \lambda_i I)^{\rho_i}(\mathbf{v}_i) = \mathbf{0}$, $i = 1, \dots, k$. Proceed as in the proof of Theorem 2.1 of Chapter 6. If $\mathbf{v}_{m+1} = c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m$, note that $\mathbf{w} = (T - \lambda_{m+1} I)^{\rho_{m+1}-1}(\mathbf{v}_{m+1})$ is an eigenvector. Using the result of part *a*, calculate $(T - \lambda_1 I)^{\rho_1}(T - \lambda_2 I)^{\rho_2} \dots (T - \lambda_m I)^{\rho_m}(\mathbf{w})$ in two ways.)

16. a. Let J be a $k \times k$ Jordan block with eigenvalue λ. Show that $(J - \lambda I)^k = O$.

b. (Cayley-Hamilton Theorem) Let A be an $n \times n$ matrix, and let $p(t)$ be its characteristic polynomial. Show that $p(A) = O$. (*Hint:* Use Theorem 1.5.)

c. Give the polynomial $q(t)$ of smallest possible degree so that $q(A) = O$. This is called the *minimal polynomial* of A. Show that $p(t)$ is divisible by $q(t)$.

17. Prove that the set of n vectors constructed in the proof of Theorem 1.5 is linearly independent. (*Hints:* We started with a linearly independent set $\{\mathbf{w}_1, \dots, \mathbf{w}_{n-d}\}$. Suppose $\sum a_k\mathbf{v}_k + \sum c_i\mathbf{w}_i + \sum d_s\mathbf{u}_s = \mathbf{0}$. Multiply by $A - \lambda I$ and check that we get only a linear combination of the $\mathbf{w}_i$, which are known to form a linearly independent set. Conclude that all the d_s and c_i, $i \neq 1, j_1 + 1, \dots, j_{\ell-1} + 1$, must be 0. This leaves only the terms involving the eigenvectors $\mathbf{w}_1, \dots, \mathbf{w}_{j_{\ell-1}+1}$, and $\mathbf{v}_{\ell+1}, \dots, \mathbf{v}_d$, but by construction these form a linearly independent set.)

2 Computer Graphics and Geometry

We have seen that linear transformations model various sorts of motions of space: rotations, reflections, shears, and even projections. But all of these motions leave the origin fixed. We also need to be able to slide objects around and look at them from different perspectives, especially if we want to implement the programming of computer graphics. To accomplish this in the context of linear transformations, we introduce the clever idea of setting $\mathbb{R}^n$ inside $\mathbb{R}^{n+1}$ as a hyperplane shifted vertically off the origin.

We begin by recalling the shear transformation defined in Example 2(b) of Section 2 of Chapter 2. If a is an arbitrary real number, then we have

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + ax_2 \\ x_2 \end{bmatrix}.$$

In particular, we calculate that the copy of the x_1-axis at height $x_2 = 1$ is transformed by the rule

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ 1 \end{bmatrix};$$

that is, we see the underlying function

$$\tau(x) = x + a$$

in action. The function τ is not linear, but we've managed to "encode" it by a linear transformation by considering the action on the line $x_2 = 1$ (rather than the x_1-axis). Such a function is called a *translation* of $\mathbb{R}$.

We can equally well consider translations in $\mathbb{R}^n$. If $\mathbf{a} \in \mathbb{R}^n$ is an arbitrary vector, we define the function

$$\tau_{\mathbf{a}} \colon \mathbb{R}^n \to \mathbb{R}^n, \quad \tau_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}.$$

Defining the matrix

$$\Psi = \left[\begin{array}{c|c} I_n & \begin{matrix} | \\ \mathbf{a} \\ | \end{matrix} \\ \hline 0 \quad \cdots \quad 0 & 1 \end{array}\right],$$

we then have

$$\Psi \begin{bmatrix} | \\ \mathbf{x} \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ \tau_{\mathbf{a}}(\mathbf{x}) \\ | \\ 1 \end{bmatrix}.$$

The schematic diagram in Figure 2.1 makes sense in higher dimensions, provided we interpret the horizontal axis as representing $\mathbb{R}^n$.

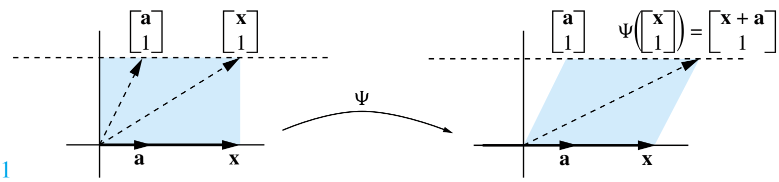

FIGURE 2.1

When we compose a linear transformation of $\mathbb{R}^n$ (given by an $n \times n$ matrix A) with a translation (given by a vector $\mathbf{a} \in \mathbb{R}^n$), we obtain an *affine transformation*, which may thus be written

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}.$$

It should come as no surprise that this transformation can be represented analogously by the $(n+1) \times (n+1)$ matrix

$$(\dagger) \qquad \Psi = \left[\begin{array}{c|c} A & \begin{matrix} | \\ \mathbf{a} \\ | \end{matrix} \\ \hline 0 \quad \cdots \quad 0 & 1 \end{array}\right].$$

As a check, note that

$$\left[\begin{array}{ccc|c} & & & | \\ & A & & \mathbf{a} \\ & & & | \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]\begin{bmatrix} | \\ \mathbf{x} \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ A\mathbf{x}+\mathbf{a} \\ | \\ 1 \end{bmatrix}.$$

Conversely, notice that any $(n+1)\times(n+1)$ matrix of the form (†) can be interpreted as an affine transformation of $\mathbb{R}^n$, defining the affine transformation f by

$$\Psi\begin{bmatrix} | \\ \mathbf{x} \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ f(\mathbf{x}) \\ | \\ 1 \end{bmatrix}.$$

EXAMPLE 1

We can represent a rotation of $\mathbb{R}^2$ through angle $\pi/3$ about the point $(1, 0)$ as a product of affine transformations. We begin by thinking about how this can be achieved geometrically: First we translate $(1, 0)$ to the origin, next we rotate an angle of $\pi/3$ about the origin, and then last, we translate the origin back to $(1, 0)$. Now we just encode each of these affine transformations in a matrix and take the product of the respective matrices:

$$B = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \hline 0 & 0 & 1 \end{array}\right].$$

The matrix on the right-hand side represents the affine transformation of $\mathbb{R}^2$ defined by a rotation by $\pi/3$ about the origin followed by a translation by the vector $\mathbf{a} = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$. It may be somewhat surprising that following a rotation about the origin by a translation is the same as rotating about some other point, but a few moments' thought will make it seem reasonable (see also Exercise 4).

Notice that the expression for B on the left-hand side looks like the change-of-basis formula in $\mathbb{R}^3$. Indeed, if we let

$$P = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad\text{and}\quad \Psi = \left[\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

then $B = P\Psi P^{-1}$. We can think of P as the change-of-basis matrix, with the old basis given by the standard basis for $\mathbb{R}^3$ and the new basis given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2, \quad\text{and}\quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(Notice that $\mathbf{v}_3$ represents the point $(1, 0)$ in the copy of $\mathbb{R}^2$ passing through $(0, 0, 1)$.) Consider the linear transformation of $\mathbb{R}^3$ that rotates the $\mathbf{v}_1\mathbf{v}_2$-plane through $\pi/3$ and leaves the vector $\mathbf{v}_3$ fixed. The matrix of this linear transformation with respect to the *new* basis is Ψ, and, by the change-of-basis formula, its matrix with respect to the *standard* basis is $B = P\Psi P^{-1}$. (It is worth emphasizing that this is very much the same way we applied the change-of-basis formula to calculate matrices for projections and rotations in Chapter 4.)

There is one last question we should answer. Suppose we'd been given the matrix B without any further information. How might we have discovered its interpretation as a rotation of $\mathbb{R}^2$ about some point? From the 2×2 matrix on the upper left we see the rotation, but how do we see the point about which we are rotating? Since that point **a** is left fixed by the affine transformation, the corresponding vector

$$\begin{bmatrix} | \\ \mathbf{a} \\ | \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

must be an eigenvector of our matrix B with corresponding eigenvalue 1. An easy computation gives the vector $\mathbf{v}_3$, as expected. ▲

EXAMPLE 2

Consider the 3×3 matrix

$$B = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \\ \hline 0 & 0 & 1 \end{array}\right].$$

We wish to analyze the affine transformation of $\mathbb{R}^2$ that B represents. (Of course, the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is quite familiar: It is the standard matrix of the reflection across the line $x_1 = x_2$. But suppose we didn't even remember this!) The eigenvalues of B are 1, 1, and -1, and we find the following bases for the eigenspaces of B:

$$\mathbf{E}(-1): \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$\mathbf{E}(1): \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

It now follows (why?) that μ_B represents a reflection about the line with direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and passing through $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as depicted in Figure 2.2.

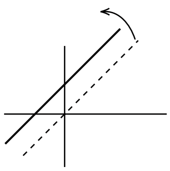

FIGURE 2.2 ▲

EXAMPLE 3

Consider next the 3×3 matrix

$$B = \left[\begin{array}{cc|c} \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

What do eigenvalues and eigenvectors tell us about B? The characteristic polynomial of B is $p(t) = -(t-1)^2(t+1)$, so the eigenvalue -1 has algebraic multiplicity 1 and the eigenvalue 1 has algebraic multiplicity 2. We determine the following bases for the eigenspaces of B:

$$\mathbf{E}(-1)\textbf{:}\quad \left\{ \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \right\}$$

$$\mathbf{E}(1)\textbf{:}\quad \left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right\}$$

Unfortunately, the geometric multiplicity of the eigenvalue 1 is only 1. But consider the vector

$$\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix};$$

we have

$$B = \left[\begin{array}{cc|c} \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{\sqrt{3}}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \frac{\sqrt{3}}{2} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

That is, $B\mathbf{v}$ is equal to the sum of $\mathbf{v}$ and a scalar multiple of the (unit) vector spanning $\mathbf{E}(1)$. With respect to the basis formed by

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix},$$

the matrix representing μ_B is

$$\Psi = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} \\ \hline 0 & 0 & 1 \end{array}\right].$$

From this we infer that μ_B gives a reflection of $\mathbb{R}^2$ about the line spanned by $\mathbf{v}_2$ and then translates by the vector $\frac{\sqrt{3}}{2}\mathbf{v}_2$. This is called a *glide reflection* with axis $\mathbf{v}_2$. ▲

2.1 Isometries of $\mathbb{R}$ and $\mathbb{R}^2$

Of particular interest are the *isometries* of $\mathbb{R}^n$: these are the functions $f: \mathbb{R}^n \to \mathbb{R}^n$ that preserve distance, i.e.,

$$\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Lemma 2.1. *Every isometry $f: \mathbb{R}^n \to \mathbb{R}^n$ can be written in the form*

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$$

for some vector $\mathbf{a} \in \mathbb{R}^n$ and some orthogonal *$n \times n$ matrix A.*

Proof. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$. Then we observe that $T(\mathbf{0}) = f(\mathbf{0}) - f(\mathbf{0}) = \mathbf{0}$, and T is also an isometry:

$$\|T(\mathbf{x}) - T(\mathbf{y})\| = \|\big(f(\mathbf{x}) - f(\mathbf{0})\big) - \big(f(\mathbf{y}) - f(\mathbf{0})\big)\| = \|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|,$$

as desired. It now follows from Exercise 4.4.23 that T is a linear transformation whose standard matrix is orthogonal. ☐

Corollary 2.2. *Every isometry of $\mathbb{R}^n$ is given by an $(n+1) \times (n+1)$ matrix of the form*

$$\left[\begin{array}{ccc|c} & & & | \\ & A & & \mathbf{a} \\ & & & | \\ \hline 0 & \cdots & 0 & 1 \end{array}\right],$$

where $\mathbf{a} \in \mathbb{R}^n$ and A is an orthogonal $n \times n$ matrix.

We can now use our experience with linear algebra to classify all isometries of $\mathbb{R}$ and $\mathbb{R}^2$. The analogous project for $\mathbb{R}^3$ is left for Exercise 13. Our first result is hardly a surprise, but it's a good warm-up for what is to follow.

Proposition 2.3. *Every isometry of $\mathbb{R}$ is either a translation or a reflection.*

Proof. Since the only orthogonal 1×1 matrices are $[1]$ and $[-1]$, it is a matter of analyzing the 2×2 matrices

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & a \\ 0 & 1 \end{bmatrix}.$$

We understand that the former represents a translation (by a) as it stands. What about the latter? Since the matrix is upper triangular, we recognize that its eigenvalues are -1 and 1, the corresponding eigenvectors being

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a/2 \\ 1 \end{bmatrix}.$$

This tells us that the isometry in question leaves the point $a/2$ fixed and reflects about it.

Note that the change-of-basis formula gives

$$\begin{bmatrix} -1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{a}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{a}{2} \\ 0 & 1 \end{bmatrix}.$$

Multiplying by $\begin{bmatrix} x \\ 1 \end{bmatrix}$ yields

$$\begin{bmatrix} -1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{a}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -(x - \frac{a}{2}) \\ 1 \end{bmatrix} = \begin{bmatrix} -(x - \frac{a}{2}) + \frac{a}{2} \\ 1 \end{bmatrix},$$

as one can check easily by elementary algebra. ☐

Life in $\mathbb{R}^2$ is somewhat more complicated. Let's begin by analyzing the isometries of $\mathbb{R}^2$ that fix the origin. These are given by orthogonal 2×2 matrices, and by Exercise 2.5.19, there are two possibilities: Either we have a rotation through angle θ, given by

(†) $$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

or we have the composition of a reflection and a rotation:

(‡) $$A = A_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

What are the eigenvalues of the latter matrix? Its characteristic polynomial is $p(t) = t^2 - 1$, and so the eigenvalues are 1 and -1. The corresponding eigenvectors are

$$\begin{bmatrix} \sin\theta \\ 1 - \cos\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\sin\theta \\ 1 + \cos\theta \end{bmatrix}.$$

We observe first that the eigenvectors are orthogonal (but of course—after all, in this case A is symmetric). Next, using the double-angle formulas

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$\cos\theta = 1 - 2\sin^2\tfrac{\theta}{2},$$

we see that the vector

$$\mathbf{v} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

forms a basis for $\mathbf{E}(1)$, and so μ_A gives rise to a reflection about the line through the origin with direction vector $\mathbf{v}$, as shown in Figure 2.3.

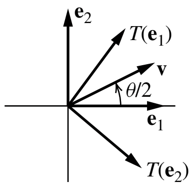

FIGURE 2.3

Theorem 2.4. *Let A be an orthogonal 2×2 matrix, $\mathbf{a} \in \mathbb{R}^2$, and*

$$\Psi = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{a} \\ \hline 0 & 0 & 1 \end{array}\right].$$

If $\det A = 1$ (i.e., is of the form (†) for some θ), then there is a matrix P of the form

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & \mathbf{b} \\ \hline 0 & 0 & 1 \end{array}\right]$$

so that $P^{-1}\Psi P$ has one of the following forms:

1. $\left[\begin{array}{cc|c} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ \hline 0 & 0 & 1 \end{array}\right]$ *(a translation)*

2. $\left[\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$ *(a rotation)* .

And if det $A = -1$ *(i.e., is of the form* (‡) *for some* θ*), then there is a matrix* P *of the form*

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{B} & \mathbf{b} \\ \hline 0 & 0 & 1 \end{array}\right],$$

where B *is a rotation matrix, so that* $P^{-1}\Psi P$ *has the one of the following forms:*

3. $\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$ *(a reflection)*

4. $\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & a \\ \hline 0 & 0 & 1 \end{array}\right]$ *(a glide reflection when* $a \neq 0$*).*

Moreover, case **3** *occurs precisely when* $a = a_1 \cos\frac{\theta}{2} + a_2 \sin\frac{\theta}{2} = 0$.

Proof. When $A = I$, we have case **1**. Next we consider what happens when

$$A = A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then 1 is an eigenvalue of the matrix Ψ with algebraic multiplicity 1. A corresponding eigenvector $\mathbf{v}_3$ must be a scalar multiple of the vector $\begin{bmatrix} | \\ \mathbf{b} \\ | \\ 1 \end{bmatrix}$ for some $\mathbf{b} \in \mathbb{R}^2$, since 1 is not an eigenvalue of A. Thus, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}_3\}$ gives a basis for $\mathbb{R}^3$, and with respect to that basis, the matrix becomes that in case **2**. (See Exercise 4.)

When A is an orthogonal matrix with det $A = -1$, then A is of the form

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some θ. This case is somewhat more interesting, as this matrix has eigenvalues -1 and 1, and hence the eigenvalues of Ψ are -1, 1, and 1. When $\mathbf{E}(1)$ is two-dimensional, we can diagonalize Ψ and therefore obtain case **3**. Now this occurs precisely when the matrix $\Psi - I$ has rank 1, and it is a straightforward computation to check that this occurs if and only if $\mathbf{a}$ is orthogonal to the vector

$$\mathbf{v} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}.$$

As the reader can check (see Exercise 6), a basis for $\mathbf{E}(1)$ is

$$\left\{ \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{a_1}{2} \\ \frac{a_2}{2} \\ 1 \end{bmatrix} \right\}.$$

And it is easy to check that a basis for $\mathbf{E}(-1)$ is

$$\left\{ \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \\ 0 \end{bmatrix} \right\}.$$

From this we see that Ψ corresponds to reflection across the line passing through $\mathbf{a}/2$ with direction vector $\mathbf{v}$. The rotation matrix B is

$$B = \begin{bmatrix} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \text{for } \alpha = \frac{\theta}{2} - \frac{\pi}{2}.$$

When $\mathbf{E}(1)$ is only one-dimensional, we leave it to the reader to check that, with respect to the same basis

$$\left\{ \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{a_1}{2} \\ \frac{a_2}{2} \\ 1 \end{bmatrix} \right\},$$

the matrix for Ψ becomes that given in case **4**, with $a = a_1 \cos\frac{\theta}{2} + a_2 \sin\frac{\theta}{2}$. Such a transformation is called a *glide reflection* because it is the composition of a reflection and a translation ("glide") by a vector parallel to the line of reflection. □

Our discussion in the proof of Theorem 2.4 establishes the following corollary.

Corollary 2.5. *Every isometry of $\mathbb{R}^2$ is either a translation, a rotation, a reflection, or a glide reflection.*

Remark. We saw that case (4) occurs when the eigenvalue 1 has algebraic multiplicity 2 but geometric multiplicity 1. In this case, as we learned in Section 1, the Jordan canonical form of A will be

$$\left[\begin{array}{c|cc} -1 & & \\ \hline & 1 & 1 \\ & 0 & 1 \end{array}\right].$$

In our case we have a, rather than 1, in the nondiagonal slot because we require that our third basis vector have a 1 as its third entry. (See Example 3.)

2.2 Perspective Projection and Projective Equivalence of Conics

Any time we "view" an object, our brain is processing some sort of image ("projection," if you will) of it on our retina. We have dealt so far with orthogonal projections ("from infinity") onto a subspace. Now we explore briefly the concept of *perspective projection*, in which we project from a fixed point (the eye) onto a plane (not containing the eye),[4] as shown in Figure 2.4.

[4] Here we consider one-eyed vision. We won't address the geometry of binocular vision, another fascinating topic.

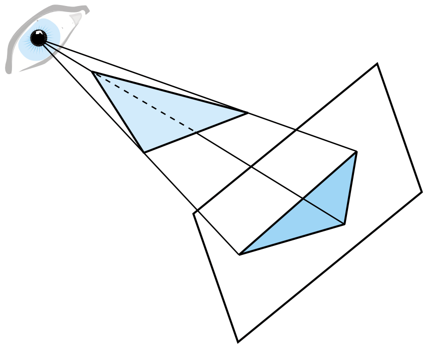

FIGURE 2.4

Given a point $\mathbf{a} \in \mathbb{R}^n$ and a hyperplane $H = \{\boldsymbol{\xi} \cdot \mathbf{x} = c\} \subset \mathbb{R}^n$, we wish to give a formula for the projection from $\mathbf{a}$ onto H, $\Pi_{\mathbf{a},H}$, as illustrated in Figure 2.5. Given a point $\mathbf{x}$, we consider the line passing through $\mathbf{a}$ and $\mathbf{x}$ and find its intersection with H. Recalling

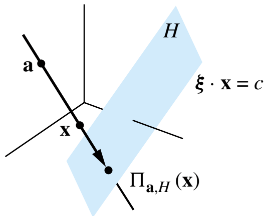

FIGURE 2.5

the parametric form of a line from Chapter 1, we then see

$$\mathbf{a} + t(\mathbf{x} - \mathbf{a}) \in H \iff \boldsymbol{\xi} \cdot \big(\mathbf{a} + t(\mathbf{x} - \mathbf{a})\big) = c,$$

and so, after a bit of calculation, we find that the projection of $\mathbf{x}$ from $\mathbf{a}$ into H is

$$(*) \qquad \Pi_{\mathbf{a},H}(\mathbf{x}) = \frac{(\boldsymbol{\xi} \cdot \mathbf{x} - c)\mathbf{a} + (c - \boldsymbol{\xi} \cdot \mathbf{a})\mathbf{x}}{\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{a})}.$$

EXAMPLE 4

Let $\mathbf{a} = \mathbf{0}$ and let $H = \{x_3 = 1\}$ in $\mathbb{R}^3$. Then

$$\Pi_{\mathbf{a},H}(\mathbf{x}) = \frac{\mathbf{x}}{x_3} = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right).$$

In case it wasn't already abundantly clear, this is decidedly *not* a linear transformation. (Note that it is undefined on points $\mathbf{x}$ with $x_3 = 0$. Why?) ▲

Earlier, we figured out how to represent affine transformations by linear transformations (one dimension higher). The question is this: Will the same trick work here? The answer is yes, provided we understand how to work with that extra dimension!

Definition. Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. We say that $\mathbf{X} = (X_1, \dots, X_n, X_{n+1}) \in \mathbb{R}^{n+1}$ is a *homogeneous coordinate vector* for $\mathbf{x}$ if

$$\frac{X_1}{X_{n+1}} = x_1, \quad \frac{X_2}{X_{n+1}} = x_2, \quad \dots, \quad \frac{X_n}{X_{n+1}} = x_n.$$

(Note, in particular, that $X_{n+1} \neq 0$ here.)

For example, $(x_1, \dots, x_n, 1)$ is a homogeneous coordinate vector for $\mathbf{x}$; this is the representation we used earlier in the section.

By means of homogeneous coordinates, we can use a linear transformation $T: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to induce a nonlinear function f (not necessarily everywhere defined) on $\mathbb{R}^n$, as follows. If $\mathbf{x} \in \mathbb{R}^n$, we take *any* homogeneous coordinate vector $\mathbf{X} \in \mathbb{R}^{n+1}$ for $\mathbf{x}$ and declare $T(\mathbf{X})$ to be a homogeneous coordinate vector of the vector $f(\mathbf{x}) \in \mathbb{R}^n$. Of course, the last entry of $T(\mathbf{X})$ needs to be nonzero for this to work.

When $T: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is nonsingular, the induced map f is called a *projective transformation* of $\mathbb{R}^n$, which, once again, may not be everywhere defined. When we add "points at infinity" that correspond to homogeneous coordinate vectors with $X_{n+1} = 0$, these transformations are the motions of a type of non-Euclidean geometry called *projective geometry*.[5] Of course, for our immediate application here, we do not expect nonsingularity, since we are interested in projections.

EXAMPLE 5

Consider the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In other words,

$$T(\mathbf{x}) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix}.$$

To determine the corresponding function f on $\mathbb{R}^3$, we consider

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix}.$$

Now, provided $x_3 \neq 0$, we can say that the latter vector is a homogeneous coordinate vector for the vector $\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right) \in \mathbb{R}^3$, which we recognize as $\Pi_{\mathbf{a},H}(\mathbf{x})$ from Example 4. ▲

[5] For more on this beautiful and classical topic, see Shifrin, *Abstract Algebra: A Geometric Approach*, Chapter 8, or Pedoe, *Geometry: A Comprehensive Course*.

EXAMPLE 6

The linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ that gives projection from the point $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ to the plane H with equation $\boldsymbol{\xi} \cdot \mathbf{x} = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = c$ can be computed with a bit of patience from our formula ($*$) above. We set

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix}, \quad \boldsymbol{\Xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ -c \end{bmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}.$$

Note that $\boldsymbol{\xi} \cdot \mathbf{x} - c = \boldsymbol{\Xi} \cdot \mathbf{X}$. If we set

$$T(\mathbf{X}) = (\boldsymbol{\Xi} \cdot \mathbf{X})\mathbf{A} - (\boldsymbol{\Xi} \cdot \mathbf{A})\mathbf{X},$$

then we have

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} | \\ (\boldsymbol{\xi} \cdot \mathbf{x} - c)\mathbf{a} + (c - \boldsymbol{\xi} \cdot \mathbf{a})\mathbf{x} \\ | \\ \boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{a}) \end{bmatrix}$$
$$= \begin{bmatrix} (c - \xi_2 a_2 - \xi_3 a_3)x_1 + \xi_2 a_1 x_2 + \xi_3 a_1 x_3 - c a_1 \\ \xi_1 a_2 x_1 + (c - \xi_1 a_1 - \xi_3 a_3)x_2 + \xi_3 a_2 x_3 - c a_2 \\ \xi_1 a_3 x_1 + \xi_2 a_3 x_2 + (c - \xi_1 a_1 - \xi_2 a_2)x_3 - c a_3 \\ \xi_1(x_1 - a_1) + \xi_2(x_2 - a_2) + \xi_3(x_3 - a_3) \end{bmatrix},$$

and, thus,

$$\Pi_{\mathbf{a},H}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \dfrac{(c - \xi_2 a_2 - \xi_3 a_3)x_1 + \xi_2 a_1 x_2 + \xi_3 a_1 x_3 - c a_1}{\xi_1(x_1 - a_1) + \xi_2(x_2 - a_2) + \xi_3(x_3 - a_3)} \\ \dfrac{\xi_1 a_2 x_1 + (c - \xi_1 a_1 - \xi_3 a_3)x_2 + \xi_3 a_2 x_3 - c a_2}{\xi_1(x_1 - a_1) + \xi_2(x_2 - a_2) + \xi_3(x_3 - a_3)} \\ \dfrac{\xi_1 a_3 x_1 + \xi_2 a_3 x_2 + (c - \xi_1 a_1 - \xi_2 a_2)x_3 - c a_3}{\xi_1(x_1 - a_1) + \xi_2(x_2 - a_2) + \xi_3(x_3 - a_3)} \end{bmatrix}.$$

It is fairly clear that computing with matrices and homogeneous coordinate vectors is superior to working with such complicated rational functions. ▲

EXAMPLE 7

The "default" view when the powerful mathematics software Mathematica draws a cube of edge 1 centered at the origin is shown in Figure 2.6. This comes from the command

```
Graphics3D[Cuboid[],Boxed -> False, Axes -> True,
AxesLabel -> {"x", "y", "z"},  ViewPoint->{1.3, -2.4, 2.0}]
```

Note that the `ViewPoint` command gives the location of the point from which we "view" the object (perhaps the point from which we project?). Another perspective is shown in Figure 2.7, using

```
ViewPoint->{4.4, 2.6, -4.0}
```

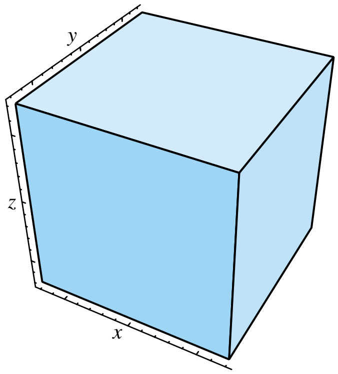

FIGURE 2.6

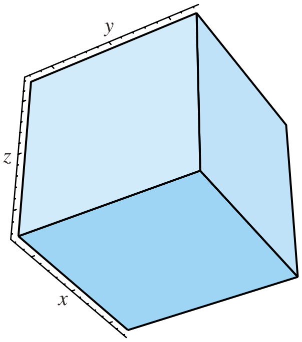

FIGURE 2.7

Of course, the folks at Mathematica don't tell us what the viewing *plane* is! To explore such issues, we refer the interested reader to Exercises 11 and 12.

Circles, ellipses, parabolas, and hyperbolas are called *conic sections* for a reason. Figure 2.8 should make that plain. Using our new tools, we can now show that these figures

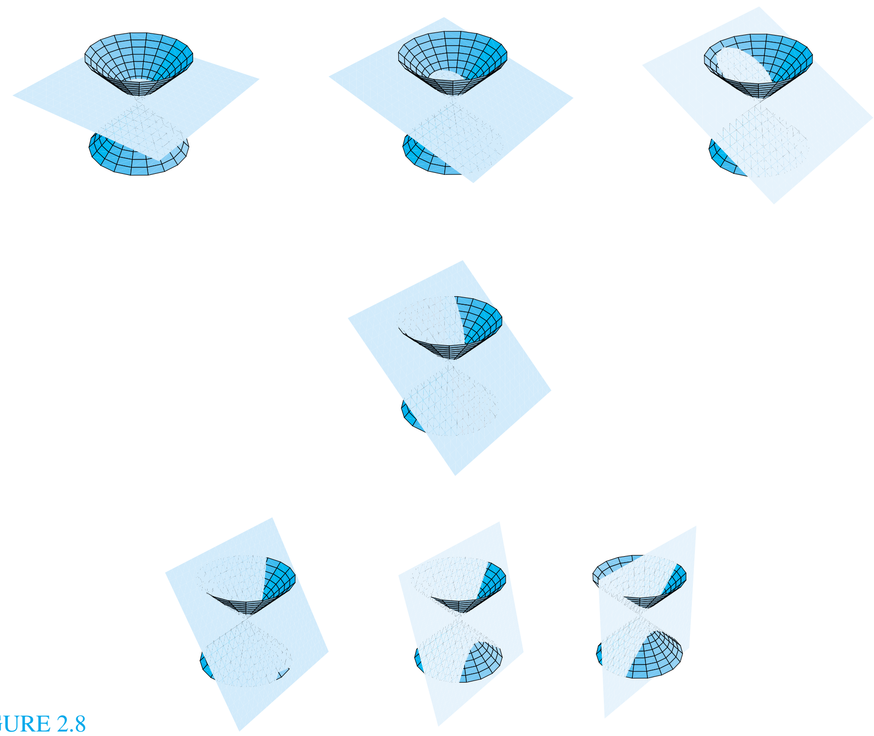

FIGURE 2.8

are all *projectively* the same. That is, with our eye at the origin in $\mathbb{R}^3$, the circle

$$x_1^2 + x_2^2 = 1, \quad x_3 = 1$$

may appear as a circle, an ellipse, a parabola, or a hyperbola, depending on what plane H we choose as our "viewing screen." Since we are projecting from the origin, which is the vertex of the cone

$$x_1^2 + x_2^2 = x_3^2,$$

the projection of the circle onto H is just the intersection of the cone with the plane H. We consider here just the family of planes

$$H_t: \quad (-\sin t)x_2 + (\cos t)x_3 = 1$$

(as illustrated in Figure 2.8). Since it is tricky to determine the equation of an intersection, we make a change of coordinates in $\mathbb{R}^3$, depending on t, so that H_t is always given by the plane $y_3 = 1$ in the new coordinates y_1, y_2, y_3. That is, let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}.$$

Then, as usual, $\mathbf{x} = Q\mathbf{y}$, and so the equation of the cone becomes

$$y_1^2 + (\cos 2t)(y_2^2 - y_3^2) - 2(\sin 2t)y_2y_3 = 0.$$

Intersecting with the plane $y_3 = 1$ gives the equation of the conic section

$$y_1^2 + (\cos 2t)y_2^2 - 2(\sin 2t)y_2 = \cos 2t.$$

As we know from Chapter 6, when $0 \le t < \pi/4$ this curve is an ellipse (indeed, a circle when $t = 0$), when $t = \pi/4$ it is a parabola, and when $\pi/4 < t \le \pi/2$ it is a hyperbola. These are pictured in Figure 2.9 for $t = 0, \pi/12, \pi/6, \pi/4, \pi/3, 5\pi/12$, and $\pi/2$, respectively.

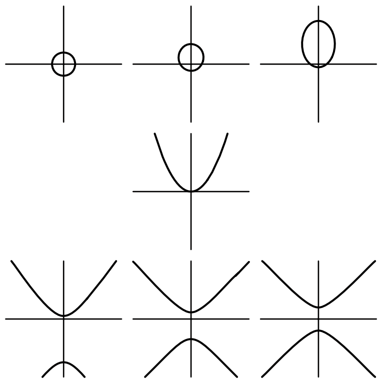

FIGURE 2.9

Remark. There is an alternative interpretation of the calculation we've just done. If we think of $\mathbf{X} = (x_1, x_2, x_3)$ as a homogeneous coordinate vector of $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, then the equation of the cone $x_1^2 + x_2^2 = x_3^2$ becomes the equation of the circle $x_1^2 + x_2^2 = 1$ when we set $x_3 = 1$. If we instead set $x_2 = 1$, then we obtain the equation of the hyperbola $-x_1^2 + x_3^2 = 1$. More interestingly, when we set $x_3 = \sqrt{2} - x_2$, we obtain the equation of a parabola, $x_1^2 - 2\sqrt{2}x_2 = 2$.

Exercises 7.2

1. In each case, give the 3×3 matrix representing the isometry of $\mathbb{R}^2$ and then use your answer to fit the isometry into our classification scheme.

*a. First translate by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and then rotate $\pi/2$ about the point $(-1, 0)$.

*b. First reflect across the line $x_1 + x_2 = 1$, and then translate by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

c. First rotate $\pi/4$ about the origin, and then reflect across the line $x_1 + x_2 = 1$.

2. Analyze each of the following isometries $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$ (according to the classification in Theorem 2.4).

a. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

*b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

e. $A = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} \sqrt{2} - 1 \\ 1 \end{bmatrix}$

3. Use techniques of linear algebra to find all the isometries of $\mathbb{R}^2$ that fix the origin and

*a. map the x_1-axis to itself.

b. map the x_1-axis to the x_2-axis.

4. Let $\theta \neq 0$. Show that

$$\Psi = \left[\begin{array}{cc|c} \cos\theta & -\sin\theta & a_1 \\ \sin\theta & \cos\theta & a_2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

represents a rotation through angle θ about the point $\frac{1}{2}\left(a_1 - a_2 \cot\frac{\theta}{2}, a_1 \cot\frac{\theta}{2} + a_2\right)$. (*Hint:* To solve for the appropriate eigenvector, you might want to use Cramer's Rule, Proposition 2.3 of Chapter 5.)

5. Let $\Psi = \left[\begin{array}{ccc|c} & & & | \\ & A & & \mathbf{a} \\ & & & | \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$. Prove that the eigenvalues of Ψ consist of the eigenvalues of A and 1.

6. Check the details in the proof of Theorem 2.4.

7. Analyze the matrices $\begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$

a. as affine transformations of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

b. as linear transformations of homogeneous coordinate vectors. From what points can we interpret these as perspective projections?

8. Prove that an affine transformation of $\mathbb{R}^2$ that leaves *three* noncollinear points fixed must be the identity. (*Hint:* Represent the affine transformation by a 3×3 matrix Ψ; use part *a* of Exercise 1.6.11 to show that the eigenvectors of Ψ are linearly independent.)

9. Given two triangles $\triangle PQR$ and $\triangle P'Q'R' \subset \mathbb{R}^2$, prove that there is an affine transformation of $\mathbb{R}^2$ carrying one to the other.

10. *a. Given two trios P, Q, R and P', Q', R' of distinct points in $\mathbb{R}$, prove there is a projective transformation f of $\mathbb{R}$ with $f(P) = P'$, $f(Q) = Q'$, and $f(R) = R'$.

b. We say three or more points in $\mathbb{R}^2$ are in general position if no three of them are ever collinear. Given two quartets P, Q, R, S and P', Q', R', S' of points in general position in $\mathbb{R}^2$, prove there is a projective transformation f of $\mathbb{R}^2$ with $f(P) = P'$, $f(Q) = Q'$, $f(R) = R'$, and $f(S) = S'$. (See Exercise 9.)

11. In this exercise we explore the problem of displaying three-dimensional images on a two-dimensional blackboard, piece of paper, or computer screen. (A computer or good graphics calculator will be helpful for experimentation here.)

a. Fix a plane $H \subset \mathbb{R}^3$ containing the x_3-axis, say $(\cos\theta)x_1 + (\sin\theta)x_2 = 0$, and fix $\mathbf{a} = (a_1, a_2, a_3) \notin H$. Find the 4×4 matrix that represents projection from $\mathbf{a}$ onto H.

b. Since we want to view on a standard screen, give a matrix that rotates H to the x_1x_2-plane, sending the x_3-axis to the x_2-axis.

c. By multiplying the two matrices you've found and deleting the row of 0's, show that the resulting linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ is given by the matrix

$$A = \begin{bmatrix} a_2 & -a_1 & 0 & 0 \\ a_3\cos\theta & a_3\sin\theta & a_1\cos\theta + a_2\sin\theta & 0 \\ \cos\theta & \sin\theta & 0 & -a_1\cos\theta - a_2\sin\theta \end{bmatrix}.$$

d. Experiment with different values of $\mathbf{a}$ and θ to obtain an effective perspective. You might compute the image of the unit cube (with one vertex at the origin and edges aligned on the coordinate axes). Pictured in Figure 2.10 are two images of the cube, first with $\mathbf{a} = (5, 4, 3)$ and $\theta = \pi/6$, next with $\mathbf{a} = (4, 6, 3)$ and $\theta = \pi/4$.

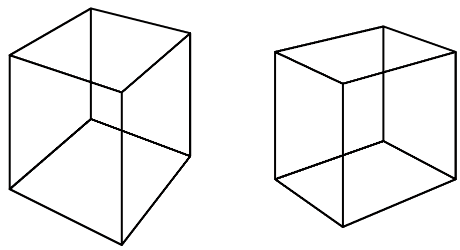

FIGURE 2.10

e. What happens if you try usual orthogonal projection onto a plane (and then rotate that plane, as before)? Show that if we take the plane with unit normal $\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$, the resulting projection $\mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$B = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\frac{1}{2}\cos\theta & -\frac{1}{2}\sin\theta & \frac{1}{2} \end{bmatrix}.$$

f. Experiment as before. For example, if we take $\theta = \pi/6$, then the image of the unit cube is as pictured in Figure 2.11. What conclusions do you reach?

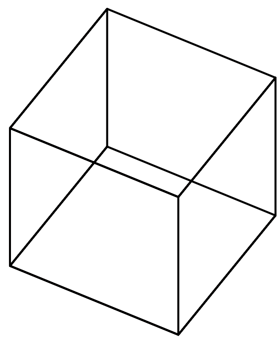

FIGURE 2.11

12. In this exercise we give the answer to the puzzle of how Mathematica draws its 3D graphics. Suppose we specify a `ViewPoint` $\mathbf{a} \in \mathbb{R}^3$ and tell Mathematica to draw an object centered at the origin. The command `ViewVertical` specifies the direction in $\mathbb{R}^3$ that "should be vertical in the final image"; the default is the usual $\mathbf{e}_3$-axis. (A computer algebra system may be helpful here.)

a. Given a "viewpoint" $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{a} \neq \mathbf{0}$, let H be the plane through the origin with normal vector $\mathbf{a}$. Find the matrix P representing the projection $\Pi_{\mathbf{a},H}$ in homogeneous coordinates.

b. Find a 4×4 matrix R that represents the rotation of $\mathbb{R}^3$ carrying H to the plane $x_3 = 0$ and carrying the "vertical direction" in H to the $\mathbf{e}_2$-axis. (*Hint:* The "vertical direction" in H should be the direction of the projection of $\mathbf{e}_3$ onto H.)

*c. Finally, by calculating the matrix RP, give the formula by which Mathematica draws the picture on the $\mathbf{e}_1\mathbf{e}_2$-plane when we specify `ViewPoint -> `$\mathbf{a}$.

d. Do some experimentation with Mathematica to convince yourself that we have solved the puzzle correctly!

13. In this exercise we analyze the isometries of $\mathbb{R}^3$.

a. If A is an orthogonal 3×3 matrix with $\det A = 1$, show that A is a rotation matrix. (See Exercise 6.2.16.) That is, prove that there is an orthonormal basis for $\mathbb{R}^3$ with respect to which the matrix takes the form

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. If A is an orthogonal 3×3 matrix with $\det A = -1$, show that there is an orthonormal basis for $\mathbb{R}^3$ with respect to which the matrix takes the form

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

That is, μ_A is the composition of a reflection across a plane with a rotation of that plane. Such a transformation is called a *rotatory reflection* when $\theta \neq 0$.

c. If A is an orthogonal 3×3 matrix and $\mathbf{a} \in \mathbb{R}^3$, prove that the matrix

$$\left[\begin{array}{ccc|c} & & & | \\ & A & & \mathbf{a} \\ & & & | \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

is similar to a matrix of one of the following forms:

$$\left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & a_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

The last such matrix corresponds to what's called a *screw* motion (why?).

d. Conclude that any isometry of $\mathbb{R}^3$ is either a rotation, a reflection, a translation, a rotatory reflection, a glide reflection, or a screw.

3 Matrix Exponentials and Differential Equations

Another powerful application of linear algebra comes from the study of systems of ordinary differential equations (ODEs). This turns out to be just a continuous version of the difference equations we studied in Section 3 of Chapter 6.

For example, in the cat/mouse problem, we used a matrix A to relate the population vector $\mathbf{x}_k$ at time k to the population vector $\mathbf{x}_{k+1}$ at time $k+1$ by the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. To think of this truly as a *difference* equation, we consider the difference

$$\mathbf{x}_{k+1} - \mathbf{x}_k = A\mathbf{x}_k - \mathbf{x}_k = (A - I)\mathbf{x}_k = \tilde{A}\mathbf{x}_k,$$

where $\tilde{A} = A - I$. If now, instead of measuring the population at discrete time intervals, we consider the population vector to be given by a differentiable function[6] of time, e.g., $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, then we get the differential equation analogue

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \text{where} \quad \frac{d\mathbf{x}}{dt} = \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}.$$

[6]Of course, the entries of a true population vector can take on only integer values, but we are taking a differentiable model that interpolates those integer values.

We can rewrite this as a system of linear differential equations:

$$\frac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t).$$

In this case, the coefficients a_{ij} are independent of t, and so we call this a constant-coefficient system of ODEs.

The main problem we address in this section is the following. Given an $n \times n$ (constant) matrix A and a vector $\mathbf{x}_0 \in \mathbb{R}^n$, we wish to find all differentiable vector-valued functions $\mathbf{x}(t)$ so that

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

(The vector $\mathbf{x}_0$ is called the *initial value* of the solution $\mathbf{x}(t)$.)

EXAMPLE 1

Suppose $n = 1$, so that $A = [a]$ for some real number a. Then we have simply the ordinary differential equation

$$\frac{dx}{dt} = ax(t), \quad x(0) = x_0.$$

The trick of "separating variables" that the reader probably learned in her integral calculus course leads to the solution[7]

$$\begin{aligned}
\frac{dx}{dt} &= ax(t) \\
\frac{dx}{x} &= a\,dt \\
\int \frac{dx}{x} &= \int a\,dt \\
\ln|x| &= at + c \\
x(t) &= Ce^{at} \qquad \text{(where we have set } C = \pm e^c\text{)}.
\end{aligned}$$

Using the fact that $x(0) = x_0$, we find that $C = x_0$ and so the solution is

$$x(t) = x_0 e^{at}.$$

As we can easily check, $\frac{dx}{dt} = ax(t)$, so we have in fact found a solution. Do we know there can be no more? Suppose $y(t)$ were any solution of the original problem. Then the function $z(t) = y(t)e^{-at}$ satisfies the equation

$$\frac{dz}{dt} = \frac{dy}{dt}e^{-at} + y(t)\left(-ae^{-at}\right) = (ay(t))\,e^{-at} + y(t)\left(-ae^{-at}\right) = 0,$$

and so $z(t)$ must be a constant function. Since $z(0) = y(0) = x_0$, we see that $y(t) = x_0e^{at}$. The original differential equation (with its initial condition) has a unique solution. ▲

[7]If the formal manipulation makes you feel uneasy, then use the chain rule to notice that $\frac{1}{x(t)}\frac{dx}{dt} = \frac{d}{dt}\ln|x(t)|$.

EXAMPLE 2

Consider perhaps the simplest possible 2×2 example:

$$\frac{dx_1}{dt} = ax_1(t)$$

$$\frac{dx_2}{dt} = \qquad bx_2(t)$$

with the initial conditions $x_1(0) = (x_1)_0$, $x_2(0) = (x_2)_0$. In matrix notation, this is the ODE

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \text{where}$$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} (x_1)_0 \\ (x_2)_0 \end{bmatrix}.$$

Since $x_1(t)$ and $x_2(t)$ appear completely independently in these equations, we infer from Example 1 that the unique solution of this system of equations will be

$$x_1(t) = (x_1)_0 e^{at}, \qquad x_2(t) = (x_2)_0 e^{bt}.$$

In vector notation, we have

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \mathbf{x}_0 = E(t)\mathbf{x}_0,$$

where $E(t)$ is the diagonal 2×2 matrix with entries e^{at} and e^{bt}. This result is easily generalized to the case of a diagonal $n \times n$ matrix. ▲

Before moving on to more complicated examples, we introduce some notation. Remember that for any positive integer k, the symbol $k!$, read "k factorial," denotes the product $k! = 1 \cdot 2 \cdot \dots \cdot (k-1) \cdot k$. By convention, $0!$ is defined to be 1. We also recall that for any real number x,

(†) $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{k!}x^k + \dots$$

(see Exercise 15). Now, given an $n \times n$ matrix A, we define a new $n \times n$ matrix e^A, called the *exponential* of A, by the "power series"

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots + \frac{1}{k!}A^k + \dots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

In general, trying to compute this series directly is extremely difficult, because the coefficients of A^k are not easily expressed in terms of the coefficients of A; indeed, it is not at all obvious that this power series will converge (but see Exercise 16). However, when A is a diagonal matrix, it is easy to compute e^A, because we know that if

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \text{then} \quad A^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix},$$

and so e^A will likewise be diagonal, with its i^{th} diagonal entry

$$\sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = e^{\lambda_i}.$$

That is,

$$\text{if} \quad A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \text{then} \quad e^A = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}.$$

Using this notation, we see that the matrix $E(t)$ that appeared in Example 2 above is just the matrix e^{tA}.

Indeed, when A is diagonalizable, there is an invertible matrix P so that $\Lambda = P^{-1}AP$ is diagonal. Thus, $A = P\Lambda P^{-1}$ and $A^k = P\Lambda^k P^{-1}$ for all positive integers k, and so

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{P\Lambda^k P^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!}\right)P^{-1} = Pe^{\Lambda}P^{-1}.$$

EXAMPLE 3

Let $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$. Then $A = P\Lambda P^{-1}$, where

$$\Lambda = \begin{bmatrix} 2 & \\ & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then we have

$$e^{t\Lambda} = \begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix} \quad \text{and} \quad e^{tA} = Pe^{t\Lambda}P^{-1} = \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{-t} & e^{-t} \end{bmatrix}.$$

▲

When $A(t)$ is a matrix-valued function of t—or, if you prefer, a matrix whose entries are functions $a_{ij}(t)$—we define the derivative, just as for vector functions above, by differentiating entry by entry:[8]

$$\frac{dA}{dt} = A'(t) = \left[a'_{ij}(t)\right].$$

The result of Example 2 generalizes to the $n \times n$ case. Indeed, as we saw in Chapter 6, whenever we can solve a problem for diagonal matrices, we can solve it for diagonalizable matrices by making the appropriate change of basis. So we should not be surprised by the following result.

---

[8]Not surprisingly, many of the usual rules of calculus have analogous matrix formulations, provided one is careful about the order of multiplication. For example, if $A(t)$ and $B(t)$ are matrices whose entries are differentiable functions of t, then $\frac{d}{dt}(A(t)B(t)) = A'(t)B(t) + A(t)B'(t)$. One can prove this entry by entry, but it is more insightful to write $A'(t) = \lim_{h\to 0}\big(A(t+h) - A(t)\big)/h$ and use the usual proof of the product rule from first-semester calculus.

Proposition 3.1. *Let A be a diagonalizable $n \times n$ matrix. The general solution of the initial value problem*

$$(*) \qquad \frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.

Proof. As above, since A is diagonalizable, there are an invertible matrix P and a diagonal matrix Λ so that $A = P\Lambda P^{-1}$ and $e^{tA} = Pe^{t\Lambda}P^{-1}$. Since the derivative of the diagonal matrix

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{bmatrix}$$

is obviously

$$\begin{bmatrix} \frac{d}{dt}e^{t\lambda_1} & & & \\ & \frac{d}{dt}e^{t\lambda_2} & & \\ & & \ddots & \\ & & & \frac{d}{dt}e^{t\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 e^{t\lambda_1} & & & \\ & \lambda_2 e^{t\lambda_2} & & \\ & & \ddots & \\ & & & \lambda_n e^{t\lambda_n} \end{bmatrix} = \Lambda e^{t\Lambda},$$

then we have

$$\begin{aligned} \frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left(Pe^{t\Lambda}P^{-1}\right) = P\left(\frac{d}{dt}e^{t\Lambda}\right)P^{-1} \\ &= P\left(\Lambda e^{t\Lambda}\right)P^{-1} \\ &= (P\Lambda P^{-1})(Pe^{t\Lambda}P^{-1}) = Ae^{tA}. \end{aligned}$$

We can now check that $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ is indeed a solution:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(e^{tA}\mathbf{x}_0\right) = (Ae^{tA})\mathbf{x}_0 = A(e^{tA}\mathbf{x}_0) = A\mathbf{x}(t),$$

as required.

Now suppose that $\mathbf{y}(t)$ is a solution of the equation $(*)$, and consider the vector function $\mathbf{z}(t) = e^{-tA}\mathbf{y}(t)$. Then, by the product rule, we have

$$\begin{aligned} \frac{d\mathbf{z}}{dt} &= \frac{d}{dt}\big(e^{-tA}\big)\mathbf{y}(t) + e^{-tA}\left(\frac{d\mathbf{y}}{dt}\right) \\ &= -Ae^{-tA}\mathbf{y}(t) + e^{-tA}\big(A\mathbf{y}(t)\big) = \big(-Ae^{-tA} + e^{-tA}A\big)\mathbf{y}(t) \\ &= \mathbf{0}, \end{aligned}$$

since $Ae^{-tA} = e^{-tA}A$ (why?). This implies that $\mathbf{z}(t)$ must be a constant vector, and so

$$\mathbf{z}(t) = \mathbf{z}(0) = \mathbf{y}(0) = \mathbf{x}_0,$$

whence $\mathbf{y}(t) = e^{tA}\mathbf{z}(t) = e^{tA}\mathbf{x}_0$ for all t, as required. □

Remark. A more sophisticated interpretation of this result is the following: If we view the system $(*)$ of ODEs in a coordinate system derived from the eigenvectors of the matrix A, then the system is uncoupled.

EXAMPLE 4

Continuing Example 3, we see that the general solution of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t)$ has the form

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{tA}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{for appropriate constants } c_1 \text{ and } c_2$$
$$= \begin{bmatrix} c_1e^{2t} \\ c_1e^{2t} - c_1e^{-t} + c_2e^{-t} \end{bmatrix} = c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_2 - c_1)e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Of course, this is the expression we get when we write

$$\mathbf{x}(t) = e^{tA}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P\left(e^{t\Lambda}P^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

and obtain the familiar linear combination of the columns of P (which are the eigenvectors of A). If, in particular, we wish to study the long-term behavior of the solution (see the discussion in Section 3 of Chapter 6), we observe that $\lim_{t\to\infty} e^{-t} = 0$ and $\lim_{t\to\infty} e^{2t} = \infty$, so that $\mathbf{x}(t)$ behaves like $c_1e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $t \to \infty$. In general, this type of analysis of diagonalizable systems is called *normal mode analysis*, and the vector functions

$$e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

corresponding to the eigenvectors are called the *normal modes* of the system. ▲

To emphasize the analogy with the solution of difference equations in Section 3 of Chapter 6 and the formula (∗) on p. 279, we rephrase Proposition 3.1 so as to highlight the normal modes.

Corollary 3.2. *Suppose A is diagonalizable, with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and write $A = P\Lambda P^{-1}$, as usual. Then the solution of the initial value problem*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 = Pe^{t\Lambda}(P^{-1}\mathbf{x}_0)$$

$$(\dagger\dagger) \qquad = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$
$$= c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_ne^{\lambda_n t}\mathbf{v}_n,$$

where

$$P^{-1}\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Note that the general solution is a linear combination of the normal modes $e^{\lambda_1 t}\mathbf{v}_1, \dots, e^{\lambda_n t}\mathbf{v}_n$.

Even when A is not diagonalizable, we may differentiate the exponential series term by term[9] to obtain

$$\frac{d}{dt}(e^{tA}) = \frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{t^k}{k!}A^k + \frac{t^{k+1}}{(k+1)!}A^{k+1} + \dots\right)$$
$$= A + tA^2 + \frac{t^2}{2!}A^3 + \cdots + \frac{t^{k-1}}{(k-1)!}A^k + \frac{t^k}{k!}A^{k+1} + \dots$$
$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1} + \frac{t^k}{k!}A^k + \dots\right) = Ae^{tA}.$$

Thus, we have the following theorem.

Theorem 3.3. *Suppose A is an $n \times n$ matrix. Then the unique solution of the initial value problem*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.

EXAMPLE 5

Consider the differential equation $\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t)$ when

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The unsophisticated (but tricky) approach is to write this system out explicitly:

$$\frac{dx_1}{dt} = -x_2(t)$$
$$\frac{dx_2}{dt} = x_1(t)$$

and differentiate again, obtaining

$$(**) \qquad \begin{aligned} \frac{d^2x_1}{dt^2} &= -\frac{dx_2}{dt} = -x_1(t) \\ \frac{d^2x_2}{dt^2} &= \frac{dx_1}{dt} = -x_2(t). \end{aligned}$$

That is, our vector function $\mathbf{x}(t)$ satisfies the second-order differential equation

$$\frac{d^2\mathbf{x}}{dt^2} = -\mathbf{x}(t).$$

Now, the equations $(**)$ have the "obvious" solutions

$$x_1(t) = a_1 \cos t + b_1 \sin t \quad \text{and} \quad x_2(t) = a_2 \cos t + b_2 \sin t$$

for some constants a_1, a_2, b_1, and b_2 (although it is far from obvious that these are the *only* solutions). Some information was lost in the process; in particular, since $\frac{dx_1}{dt} = x_2$, the constants must satisfy the equations

$$a_2 = -b_1 \quad \text{and} \quad b_2 = a_1.$$

[9] One cannot always differentiate infinite sums term by term, but it is proved in an analysis course that it is valid to do so with power series such as this.

That is, the vector function

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a\cos t - b\sin t \\ a\sin t + b\cos t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

gives a solution of the original differential equation.

On the other hand, Theorem 3.3 tells us that the general solution should be of the form

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0,$$

and so we suspect that

$$e^{\left(t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

should hold. Well,

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \end{bmatrix}. \end{aligned}$$

Since the power series expansions (Taylor series) for sine and cosine are, indeed,

$$\begin{aligned} \sin t &= t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots + (-1)^k \frac{1}{(2k+1)!}t^{2k+1} + \dots \\ \cos t &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots + (-1)^k \frac{1}{(2k)!}t^{2k} + \dots, \end{aligned}$$

the formulas agree.

Another approach to computing e^{tA} is to diagonalize A over the complex numbers (the first part of Section 1 is needed here). The characteristic polynomial of A is $p(t) = t^2 + 1$, with roots $\pm i$. That is, the eigenvalues of A are i and $-i$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix},$$

as the reader can easily check. That is,

$$A = P\Lambda P^{-1}, \quad \text{where} \quad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{tA} = Pe^{t\Lambda}P^{-1} &= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ -i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{bmatrix}. \end{aligned}$$

Now comes one of the great mathematical relationships of all time, the discovery of which is usually attributed to Euler:[10] If we substitute it for x in the equation (†) on p. 333, we t obtain

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = 1 + it - \frac{1}{2!}t^2 - i\frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \dots$$
$$= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots\right) + i\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right) = \cos t + i \sin t.$$

Then it follows immediately that

$$e^{tA} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ -i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

exactly as before. ▲

EXAMPLE 6

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix},$$

whose characteristic polynomial is $p(t) = t^2 - 2t + 5$. Thus, the eigenvalues of A are $1 \pm 2i$, with respective eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

The general solution of the differential equation $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{(1+2i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-2i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$= c_1 e^t(\cos 2t + i \sin 2t)\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^t(\cos 2t - i \sin 2t)\begin{bmatrix} 1 \\ -i \end{bmatrix},$$

and, separating this expression into its real and imaginary parts, we obtain

$$= (c_1 + c_2)e^t\begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + i(c_1 - c_2)e^t\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$
$$= e^t\begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}\begin{bmatrix} c_1 + c_2 \\ i(c_1 - c_2) \end{bmatrix}.$$

[10] Although Euler published this in 1743, the result was apparently first discovered by Cotes in 1714. See Maor, *e: The Story of a Number*.

Since the matrix A is real, the real and imaginary parts of $\mathbf{x}(t)$ must be solutions, and indeed, the general solution is a linear combination of the *normal modes*

$$e^t \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} \quad \text{and} \quad e^t \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}.$$

Solution curves are spirals emanating from the origin (as $t \to -\infty$) with exponentially increasing radius. ▲

EXAMPLE 7

Let's now consider the case of a non-diagonalizable matrix, such as

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

The system

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + x_2 \\ \frac{dx_2}{dt} &= \qquad 2x_2 \end{aligned}$$

is already partially uncoupled, so we know that $x_2(t)$ must take the form $x_2(t) = ce^{2t}$ for some constant c. Now, in order to find $x_1(t)$, we must solve the inhomogeneous ODE

$$\frac{dx_1}{dt} = 2x_1(t) + ce^{2t}.$$

In elementary differential equations courses, one is taught to look for a solution of the form

$$x_1(t) = ae^{2t} + bte^{2t};$$

in this case,

$$\frac{dx_1}{dt} = (2a + b)e^{2t} + (2b)te^{2t} = 2x_1(t) + be^{2t},$$

and so taking $b = c$ gives the desired solution to our equation. That is, the solution to the system is the vector function

$$\mathbf{x}(t) = \begin{bmatrix} ae^{2t} + cte^{2t} \\ ce^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

The explanation of the trick is quite simple. Let's calculate the matrix exponential e^{tA} by writing

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2I + B, \quad \text{where} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The powers of A are easy to compute because $B^2 = 0$: By the binomial theorem (see Exercise 2.1.15),

$$(2I + B)^k = 2^k I + k2^{k-1} B,$$

and so

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(2^k I + k 2^{k-1} B\right) \\ &= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} I + \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(k 2^{k-1}\right) B \\ &= e^{2t} I + t \sum_{k=1}^{\infty} \frac{(2t)^{k-1}}{(k-1)!} B = e^{2t} I + t \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} B \\ &= e^{2t} I + t e^{2t} B = \begin{bmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

A similar phenomenon occurs with any matrix in Jordan canonical form (see Exercises 4 and 8). ▲

Let's consider the general n^{th}-order linear ODE with constant coefficients:

$$(\star) \qquad y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0.$$

Here $a_0, a_1, \dots, a_{n-1}$ are scalars, and $y(t)$ is assumed to be n-times differentiable; $y^{(k)}(t)$ denotes its k^{th} derivative. We can use the power of Theorem 3.3 to derive the following general result. (See Section 6 of Chapter 3 for a discussion of the vector space $\mathcal{C}^\infty(\mathcal{I})$ of infinitely differentiable functions on an interval $\mathcal{I}$.)

Theorem 3.4. *Let n be a positive integer. The set of solutions of the n^{th}-order ODE $(\star)$ is an n-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R})$, the vector space of infinitely differentiable functions defined on $\mathbb{R}$. In particular, the initial value problem*

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0$$
$$y(0) = c_0, \quad y'(0) = c_1, \quad y''(0) = c_2, \quad \dots, \quad y^{(n-1)}(0) = c_{n-1}$$

has a unique solution.

Proof. The trick is to concoct a way to apply Theorem 3.3. We introduce the vector function $\mathbf{x}(t)$ defined by

$$\mathbf{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

and observe that it satisfies the *first*-order system of ODEs

$$\begin{aligned} \frac{d\mathbf{x}}{dt} = \begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \\ \vdots \\ y^{(n)}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} \\ &= A\mathbf{x}(t), \end{aligned}$$

where A is the obvious matrix of coefficients. We infer from Theorem 3.3 that the general solution is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, so

$$\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = e^{tA} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = c_0\mathbf{v}_1(t) + c_1\mathbf{v}_2(t) + \cdots + c_{n-1}\mathbf{v}_n(t),$$

where $\mathbf{v}_j(t)$ are the columns of e^{tA}. In particular, if we let $q_1(t), \dots, q_n(t)$ denote the first entries of the vector functions $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$, respectively, we see that

$$y(t) = c_0q_1(t) + c_1q_2(t) + \cdots + c_{n-1}q_n(t);$$

that is, the functions $q_1, \dots, q_n$ span the vector space of solutions of the differential equation ($\star$). Note that these functions are infinitely differentiable since the entries of e^{tA} are. Last, we claim that these functions are linearly independent. Suppose that for some scalars $c_0, c_1, \dots, c_{n-1}$, we have

$$y(t) = c_0q_1(t) + c_1q_2(t) + \cdots + c_{n-1}q_n(t) = 0 \text{ for all } t.$$

Then, differentiating, we have the same linear relation among all of the k^{th} derivatives of $q_1, \dots, q_n$, for $k = 1, \dots, n-1$, and so we have

$$e^{tA} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = c_0\mathbf{v}_1(t) + c_1\mathbf{v}_2(t) + \cdots + c_{n-1}\mathbf{v}_n(t) = \mathbf{0}.$$

Since e^{tA} is an invertible matrix (see Exercise 13), we infer that $c_0 = c_1 = \cdots = c_{n-1} = 0$, and so $\{q_1, \dots, q_n\}$ is linearly independent. □

EXAMPLE 8

Let

$$A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix},$$

and consider the *second*-order system of ODEs

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \frac{d\mathbf{x}}{dt}(0) = \mathbf{x}_0'.$$

The experience we gained in Example 5 suggests that if we can uncouple this system (by finding eigenvalues and eigenvectors), we should expect to find normal modes that are sinusoidal in nature.

The characteristic polynomial of A is $p(t) = t^2 + 6t + 5$, and so its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -5$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(Note, as a check, that because A is symmetric, the eigenvectors are orthogonal.) As usual, we write $P^{-1}AP = \Lambda$, where

$$\Lambda = \begin{bmatrix} -1 & \\ & -5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Let's make the "uncoupling" change of coordinates $\mathbf{y} = P^{-1}\mathbf{x}$, i.e.,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then the system of differential equations becomes

$$\frac{d^2\mathbf{y}}{dt^2} = P^{-1}\frac{d^2\mathbf{x}}{dt^2} = P^{-1}A\mathbf{x} = \Lambda P^{-1}\mathbf{x} = \Lambda\mathbf{y},$$

i.e.,

$$\begin{aligned} \frac{d^2y_1}{dt^2} &= -y_1 \\ \frac{d^2y_2}{dt^2} &= \qquad -5y_2, \end{aligned}$$

whose general solution is

$$\begin{aligned} y_1(t) &= a_1\cos t + b_1\sin t \\ y_2(t) &= a_2\cos\sqrt{5}t + b_2\sin\sqrt{5}t. \end{aligned}$$

This means that in the original coordinates, we have $\mathbf{x} = P\mathbf{y}$, i.e.,

$$\begin{aligned} \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1\cos t + b_1\sin t \\ a_2\cos\sqrt{5}t + b_2\sin\sqrt{5}t \end{bmatrix} \\ &= (a_1\cos t + b_1\sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (a_2\cos\sqrt{5}t + b_2\sin\sqrt{5}t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

The four constants can be determined from the initial conditions $\mathbf{x}_0$ and $\mathbf{x}_0'$. For example, if we start with

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0' = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $a_1 = a_2 = \frac{1}{2}$ and $b_1 = b_2 = 0$. Note that the form of our solution looks very much like the normal mode decomposition of the solution (††) of the first-order system on p. 336.

A physical system that leads to this differential equation is the following. Hooke's Law says that a spring with spring constant k exerts a restoring force $F = -kx$ on a mass m that is displaced x units from its equilibrium position (corresponding to the "natural length" of the spring). Now imagine a system, as pictured in Figure 3.1, consisting of two masses (m_1 and m_2) connected to each other and to walls by three springs (with spring constants k_1, k_2, and k_3). Denote by x_1 and x_2 the displacement of masses m_1 and m_2, respectively, from

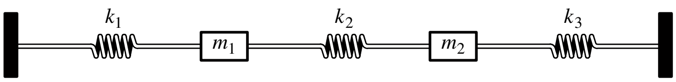

FIGURE 3.1

equilibrium position. Hooke's Law, as stated above, and Newton's second law of motion ("force = mass × acceleration") give us the following system of equations:

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= -k_1x_1 + k_2(x_2 - x_1) = -(k_1+k_2)x_1 + k_2x_2 \\ m_2\frac{d^2x_2}{dt^2} &= k_2(x_1 - x_2) - k_3x_2 = k_2x_1 - (k_2+k_3)x_2. \end{aligned}$$

Setting $m_1 = m_2 = 1$, $k_1 = k_3 = 1$, and $k_2 = 2$ gives the system of differential equations with which we began. Here the normal modes correspond to sinusoidal motion with $x_1 = x_2$ (so we observe the masses moving "in parallel," the middle spring staying at its natural length) and frequency 1 and to sinusoidal motion with $x_1 = -x_2$ (so we observe the masses moving "in antiparallel," the middle spring compressing symmetrically) and frequency $\sqrt{5}$. The general solution is a superposition of these two motions.

In Exercise 11 we ask the reader to solve this problem by converting it to a system of *first*-order differential equations, as in the proof of Theorem 3.4. ▲

Exercises 7.3

1. Calculate e^{tA} and use your answer to solve $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

*a. $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

*d. $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

*e. $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

f. $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}$

2. Solve $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, $\frac{d\mathbf{x}}{dt}(0) = \mathbf{x}_0'$.

*a. $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$, $\mathbf{x}_0' = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\mathbf{x}_0' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{x}_0' = \begin{bmatrix} 2-3\sqrt{2} \\ 2+3\sqrt{2} \end{bmatrix}$

*d. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_0' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3. Find the motion of the two-mass, three-spring system in Example 8 when

a. $m_1 = m_2 = 1$ and $k_1 = k_3 = 1$, $k_2 = 3$

b. $m_1 = m_2 = 1$ and $k_1 = 1$, $k_2 = 2$, $k_3 = 4$

*c. $m_1 = 1$, $m_2 = 2$, $k_1 = 1$, and $k_2 = k_3 = 2$

*4. Let

$$J = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}.$$

Calculate e^{tJ}.

*5. By mimicking the proof of Theorem 3.4, convert the following second-order differential equations into first-order systems and use matrix exponentials to solve them.

a. $y''(t) - y'(t) - 2y(t) = 0$, $y(0) = -1$, $y'(0) = 4$

b. $y''(t) - 2y'(t) + y(t) = 0$, $y(0) = 1$, $y'(0) = 2$

6. Check that if A is an $n \times n$ matrix and the $n \times n$ differentiable matrix function $E(t)$ satisfies $\frac{dE}{dt} = AE(t)$ and $E(0) = I$, then $E(t) = e^{tA}$ for all $t \in \mathbb{R}$.

7. Verify that $\frac{d}{dt}\sin t = \cos t$ and $\frac{d}{dt}\cos t = -\sin t$ by differentiating the power series expansions of sin and cos.

8. a. Consider the $n \times n$ matrix

$$B = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

Calculate $B^2, B^3, \dots, B^n$. (*Hint:* $B^n = \mathrm{O}$.)

b. Let J be an $n \times n$ Jordan block with eigenvalue λ. Show that

$$e^{tJ} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} & \cdots & \frac{1}{(n-1)!}t^{n-1}e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} & \cdots & \frac{1}{(n-2)!}t^{n-2}e^{\lambda t} \\ & & \ddots & \ddots & \vdots \\ & & & e^{\lambda t} & te^{\lambda t} \\ & & & & e^{\lambda t} \end{bmatrix}.$$

(*Hint:* Write $J = \lambda I + B$, and use Exercise 2.1.15 to find J^k.)

9. Use the results of Exercise 8 and Theorem 3.4 to give the general solution of the differential equations:

a. $y''(t) - 2\lambda y'(t) + \lambda^2 y(t) = 0$

b. $y'''(t) - 3\lambda y''(t) + 3\lambda^2 y'(t) - \lambda^3 y(t) = 0$

c. $y^{(4)} - 4\lambda y'''(t) + 6\lambda^2 y''(t) - 4\lambda^3 y'(t) + \lambda^4 y(t) = 0$

10. Let $a, b \in \mathbb{R}$. Convert the constant coefficient second-order differential equation

$$y''(t) + ay'(t) + by(t) = 0$$

into a first-order system by letting $\mathbf{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Considering separately the cases $a^2 - 4b \neq 0$ and $a^2 - 4b = 0$, use matrix exponentials to find the general solution.

11. By introducing the vector function

$$\mathbf{z}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_1'(t) \\ x_2'(t) \end{bmatrix},$$

show that the second-order system $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}(t)$ in Example 8 can be expressed as a first-order system $\frac{d\mathbf{z}}{dt} = B\mathbf{z}(t)$, where

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of B, calculate e^{tB}, and solve the original problem. (*Hint:* Part *c* of Exercise 5.1.9 gives a slick way to calculate the characteristic polynomial of B, but it's not too hard to do so directly.)

12. Find the solutions of the systems $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}(t)$ in Exercise 2 by converting them to first-order systems, as in Exercise 11.

13. Let A be a square matrix.
a. Prove that $Ae^{tA} = e^{tA}A$.
b. Prove that $(e^A)^{-1} = e^{-A}$. (*Hint:* Differentiate the product $e^{tA}e^{-tA}$.)
c. Prove that if A is skew-symmetric (i.e., $A^{\mathsf{T}} = -A$), then e^A is an orthogonal matrix.

14. Prove that $\det(e^A) = e^{\text{tr}A}$. (*Hint:* First assume A is diagonalizable. In the general case, apply the result of Exercise 6.2.15, which also works with complex matrices.)

15. (For those who've thought about convergence issues) Check that the power series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for any real number x and that $f(x) = e^x$, as follows.
a. Fix $x \neq 0$ and choose an integer K so that $K \geq 2|x|$. Then show that for $k > K$, we have $\frac{|x|^k}{k!} \leq C\left(\frac{1}{2}\right)^{k-K}$, where $C = \frac{|x|}{K} \cdot \ldots \cdot \frac{|x|}{2} \cdot \frac{|x|}{1}$ is a fixed constant.
b. Conclude that the series $\sum\limits_{k=K+1}^{\infty} \frac{|x|^k}{k!}$ is bounded by the convergent geometric series $C\sum\limits_{j=1}^{\infty} \frac{1}{2^j}$ and therefore converges and, thus, that the entire original series converges absolutely.
c. It is a fact that every convergent power series may be differentiated (on its interval of convergence) term by term to obtain the power series of the derivative (see Spivak, *Calculus*, Chapter 24). Check that $f'(x) = f(x)$ and deduce that $f(x) = e^x$.

16. (For those who've thought about convergence issues) Check that the power series expansion for e^A converges for any $n \times n$ matrix, as follows. (Thinking of the vector space $\mathcal{M}_{n\times n}$ of $n \times n$ matrices as $\mathbb{R}^{n^2}$ makes what follows less mysterious.)
a. If $A = \left[a_{ij}\right]$, set $\|A\| = \sqrt{\sum\limits_{i,j=1}^{n} a_{ij}^2}$. Prove that
(i) $\|cA\| = |c|\|A\|$ for any scalar c
(ii) $\|A + B\| \leq \|A\| + \|B\|$ for any $A, B \in \mathcal{M}_{n\times n}$

(iii) $\|AB\| \leq \|A\|\|B\|$ for any $A, B \in \mathcal{M}_{n\times n}$. (*Hint:* Express the entries of the matrix product in terms of the row vectors $\mathbf{A}_i$ and the column vectors $\mathbf{b}_j$.) In particular, deduce that $\|A^k\| \leq \|A\|^k$ for all positive integers k.

b. It is a fact from analysis that if $\mathbf{v}_k \in \mathbb{R}^N$ is a sequence of vectors in $\mathbb{R}^N$ with the property that $\sum_{k=1}^{\infty} \|\mathbf{v}_k\|$ converges (in $\mathbb{R}$), then $\sum_{k=1}^{\infty} \mathbf{v}_k$ converges (in $\mathbb{R}^N$). Using this fact, prove that $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges for any matrix $A \in \mathcal{M}_{n\times n}$.

c. (For those who know what a Cauchy sequence is) Prove the fact stated in part *b*.

HISTORICAL NOTES

The Jordan of the Jordan canonical form is not Wilhelm Jordan, mentioned in earlier historical notes, but Camille Jordan (1838–1922), a brilliant French mathematician. Jordan was interested in algebra and its application to geometry. In particular, he studied algebraic objects called *groups*, which are used to study symmetry—in Jordan's case, the structure of crystals. In 1870, Jordan published his most important work, a summary of group theory and related algebraic notions, in which he introduced the "canonical form" for matrices of transformations that now bears his name. At the time, mathematicians were very active in many countries. Given the lack of modern communication methods, it was not uncommon for someone to "discover" a result that had already been discovered. The history is fuzzy, but a number of different people, including Karl Weierstrass (1815–1897), Henry Smith (1826–1883), and Hermann Grassmann (1809–1877), might also be given credit. Ferdinand Frobenius (1849–1917), publishing after Jordan, explained the Jordan canonical form in its most general terms.

In this chapter you also encountered a contemporary application of linear algebra to projection on a computer screen or, more generally speaking, to the concept of perspective. Questions about perspective date back to the ancient Greeks. Euclid (ca. 325–265 BCE), in one of his many lasting works, *Optics*, raised numerous questions on perspective, wondering how simple geometric objects such as a circle appear when viewed from different planes.

Later discourses on perspective can be found during the fifteenth and sixteenth centuries, but not from mathematicians. The painter Leonardo Da Vinci (1452–1519) thought of painting as a projection of the three-dimensional world onto a two-dimensional world and sought the best way to perform this "mapping." The Italian architect Fillipo Brunelleschi (1377–1446) formulated perspective in a mathematical way and defined the concept of the "vanishing point," that place where parallel lines meet in one's view. It was the German scientist and astronomer Johannes Kepler (1571–1630) who adopted the interpretation that there was a point at infinity through which lines could be drawn.

Kepler's idea paved the way for a mathematical point of view of perspective and projection, leading to the field called projective geometry. His work led to study by Girard Desargues (1591–1661), René Descartes (1597–1650), and Étienne Pascal (1588–1651) and his son, Blaise (1623–1662). The field then lay dormant until the early nineteenth century, at which time Jean-Victor Poncelet (1788–1867) did seminal work on projective duality.

FOR FURTHER READING

More on Linear Algebra

Bretscher, Otto, *Linear Algebra with Applications*, Third Edition, Prentice Hall, 2004. A bit more depth on dynamical systems, discrete and continuous.

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Lawson, Terry, *Linear Algebra*, John Wiley & Sons, 1996. A book comparable in level to, but slightly more difficult than, this text. More details on complex vector spaces and discussion of the geometry of orthogonal matrices.

Sadun, Lorenzo, *Applied Linear Algebra: The Decoupling Principle*, Second Edition, American Mathematical Society, 2008. This book, along with Strang's *Introduction to Applied Mathematics*, delves deeply into Fourier series and differential equations, including a fair amount of infinite-dimensional linear algebra.

Strang, Gilbert, *Introduction to Linear Algebra*, Fourth Edition, Wellesley-Cambridge Press, 2009. A book more elementary than this, with more emphasis on numerical applications and less on definitions and proofs.

———, *Linear Algebra and Its Applications*, Fourth Edition, Saunders, 2008. The classic, with far more depth on applications, and the inspiration for our brief section on graph theory.

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Other Interesting Sources

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Pedoe, Dan, *Geometry: A Comprehensive Course*, Dover Publications, 1988 (originally published by Cambridge University Press, 1970). A fabulous, linear-algebraic treatment of geometry, both Euclidean and non-Euclidean, with an excellent treatment of projective geometry and quadrics.

Shifrin, Theodore, *Abstract Algebra: A Geometric Approach*, Prentice Hall, 1996. A first course in abstract algebra that will be accessible to anyone who has enjoyed this linear algebra course.

———, *Multivariable Mathematics: Linear Algebra, Multivariable Calculus, and Manifolds*, John Wiley & Sons, 2004. An integrated treatment of the linear algebra in this course and rigorous multivariable calculus. The derivative as the linearization is apparent throughout; a detailed treatment of determinants and n-dimensional volume and the change-of-variables theorem.

Spivak, Michael, *Calculus*, Fourth Edition, Publish or Perish, 2008. The ultimate source for calculus "done right."

Strang, Gilbert, *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, 1986. Although not a great source for the details and proofs, this book is a wonderful exposition of modern applied mathematics, in which the author emphasizes how even the differential equations problems follow the models established by linear algebra.

ANSWERS TO SELECTED EXERCISES

1.1.3 (4, 3, 7), (0, 5, −1), (2, −1, 3)

1.1.5 **a.**, **b.** yes; **c.** no

1.1.6 **b.** $\mathbf{x} = (-1, 2) + t(3, 1)$; **f.** $\mathbf{x} = (1, 2, 1) + t(1, -1, -1)$; **h.** $\mathbf{x} = (1, 1, 0, -1) + t(1, -2, 3, -1)$

1.1.8 **a.** no; **b.**, **c.** yes

1.1.9 **a.**, **c.** yes; **b.**, **d.** no

1.1.10 **b.** $\mathbf{x} = (1, 1, 1) + s(-3, 0, 1) + t(1, 3, 1)$

1.1.12 The planes $\mathcal{P}_1$ and $\mathcal{P}_4$ are the same. Note that $(0, 2, 1) = (1, 1, 0) + 1(1, 0, 1) + 1(-2, 1, 0)$; both vectors $(1, -1, -1)$ and $(3, -1, 1)$ are in the plane spanned by $(1, 0, 1)$ and $(-2, 1, 0)$. Thus, every point of $\mathcal{P}_4$ lies in the plane $\mathcal{P}_1$. On the other hand, $(1, 1, 0) = (0, 2, 1) + 1(1, -1, -1) + 0(3, -1, 1)$, and both vectors $(1, 0, 1)$ and $(-2, 1, 0)$ are in the plane spanned by $(1, -1, -1)$ and $(3, -1, 1)$. So every point of $\mathcal{P}_1$ lies in the plane $\mathcal{P}_4$. This means that $\mathcal{P}_1 = \mathcal{P}_4$. Similarly, $\mathcal{P}_2 = \mathcal{P}_3$.

1.1.15 The battle plan is to let $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AC} = \mathbf{y}$ and then to express $\overrightarrow{AE}$ and $\overrightarrow{AQ}$ as linear combinations of $\mathbf{x}$ and $\mathbf{y}$. We are given the facts that $\overrightarrow{AD} = \frac{2}{3}\mathbf{x}$ and $\overrightarrow{CE} = \frac{2}{5}\overrightarrow{CB} = \frac{2}{5}(\mathbf{x} - \mathbf{y})$. Therefore, $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE} = \mathbf{y} + \frac{2}{5}(\mathbf{x} - \mathbf{y}) = \frac{2}{5}\mathbf{x} + \frac{3}{5}\mathbf{y} = \frac{1}{5}(2\mathbf{x} + 3\mathbf{y})$. On the other hand, because Q is the midpoint of $\overline{CD}$, we have $\overrightarrow{AQ} = \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CD} = \mathbf{y} + \frac{1}{2}\left(\frac{2}{3}\mathbf{x} - \mathbf{y}\right) = \frac{1}{2}\mathbf{y} + \frac{1}{3}\mathbf{x} = \frac{1}{6}(2\mathbf{x} + 3\mathbf{y})$. Comparing the final expressions for $\overrightarrow{AE}$ and $\overrightarrow{AQ}$, we see that $\overrightarrow{AQ} = \frac{5}{6}\overrightarrow{AE}$, so $c = 5/6$.

1.1.23 Suppose that ℓ and $\mathcal{P}$ intersect; then there must be a point, $\mathbf{x}$, contained in both. This means that there are real numbers r, s, and t satisfying $\mathbf{x} = \mathbf{x}_0 + r\mathbf{v} = s\mathbf{u} + t\mathbf{v}$. Then $\mathbf{x}_0 = s\mathbf{u} + t\mathbf{v} - r\mathbf{v} = s\mathbf{u} + (t - r)\mathbf{v}$, so $\mathbf{x}_0 \in \text{Span}(\mathbf{u}, \mathbf{v})$, whence $\mathbf{x}_0 \in \mathcal{P}$.

1.2.1 **c.** -25, $\theta = \arccos(-5/13)$; **f.** 2, $\theta = \arccos(1/5)$

1.2.2 **c.** $-\frac{5}{13}(7, -4), -\frac{5}{13}(1, 8)$; **f.** $(-1, 0, 1), \frac{1}{25}(3, -4, 5)$

1.2.4 $\arccos\sqrt{2/3} \approx 0.62$ radians $\approx 35.3°$

1.2.6 Since $\theta = \arccos(-1/6)$, we have $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta = -1$.
Then $(\mathbf{x} + 2\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \|\mathbf{x}\|^2 - \mathbf{x} \cdot \mathbf{y} + 2(\mathbf{y} \cdot \mathbf{x}) - 2\|\mathbf{y}\|^2 = 9 + 1 - 2 - 8 = 0$, so $(\mathbf{x} + 2\mathbf{y})$ and $(\mathbf{x} - \mathbf{y})$ are orthogonal, by definition.

1.2.10 $\pi/6$

1.2.14 Let $\mathbf{x} = \overrightarrow{CA}$ and $\mathbf{y} = \overrightarrow{CB}$. Then $\overrightarrow{AB} = \mathbf{y} - \mathbf{x}$, and

$$\|\overrightarrow{AB}\|^2 = \|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{y}\|^2 - 2\mathbf{y} \cdot \mathbf{x} + \|\mathbf{x}\|^2 = a^2 - 2ab\cos\theta + b^2 .$$

1.3.1 **b.** $x_1 - x_2 + x_3 = 1$; **d.** $x_1 - 2x_2 + x_3 = 1$; **f.** $x_1 + x_2 + x_3 + x_4 = 0$

1.3.2 **a.**, **d.** $x_1 + 2x_2 - x_3 = 3$; **b.**, **c.** $x_1 + 2x_2 - x_3 = 2$

1.3.3 **c.** $\mathbf{x} = (5, 0, 0, 0) + x_2(-1, 1, 0, 0) + x_3(1, 0, 1, 0) + x_4(-2, 0, 0, 1)$; **d.** $\mathbf{x} = (4, 0, 0, 0) + x_2(2, 1, 0, 0) + x_3(-3, 0, 1, 0) + x_4(0, 0, 0, 1)$

1.3.4 **c.** $1/\sqrt{3}$; **e.** $2/9$

1.3.6 **a.** $\mathbf{x} = x_2(-5, 1, 0) + x_3(2, 0, 1)$; **b.** $(3, 0, 0)$, $\mathbf{x} = (3, 0, 0) + x_2(-5, 1, 0) + x_3(2, 0, 1)$; **c.** $\mathbf{x} = x_2(-5, 1, 0, 0) + x_3(2, 0, 1, 0) + x_4(-1, 0, 0, 1)$, $\mathbf{x} = (3, 0, 0, 0) + x_2(-5, 1, 0, 0) + x_3(2, 0, 1, 0) + x_4(-1, 0, 0, 1)$

1.3.7 **a.** $\mathbf{a} = (2, -3)$. **b.** $|c|/\|\mathbf{a}\| = 5/\sqrt{13}$. Remember that this comes from choosing a point $\mathbf{x}_0$ on the line, say $\mathbf{x}_0 = (1, -1)$, and projecting the vector $\mathbf{x}_0$ onto the normal vector $\mathbf{a}$. **c.** The line through $\mathbf{0}$ with direction vector $\mathbf{a}$ has parametric equation $\mathbf{x} = t(2, -3)$. This line intersects the given line when $2(2t) - 3(-3t) = 5$, i.e., when $t = 5/13$. Thus, $\mathbf{p} = \frac{5}{13}(2, -3)$ is the point on the line closest to the origin. $\|\mathbf{p}\| = 5/\sqrt{13}$ checks with our answer to part *b*. **d.** We choose a point $\mathbf{x}_0 = (1, -1)$ on the line and find the length of the projection of $\mathbf{x}_0 - \mathbf{w}$ onto the normal vector $\mathbf{a}$: $\|\text{proj}_{\mathbf{a}}(\mathbf{x}_0 - \mathbf{w})\| = |\mathbf{a} \cdot (\mathbf{x}_0 - \mathbf{w})|/\|\mathbf{a}\| = 2/\sqrt{13}$. **e.** The line through $\mathbf{w}$ with direction vector $\mathbf{a}$ has parametric equation $\mathbf{x} = (3, 1) + t(2, -3)$. This line intersects the given line when $2(3 + 2t) - 3(1 - 3t) = 5$, i.e., when $t = 2/13$. This gives the point $\mathbf{q} = \mathbf{w} + \frac{2}{13}\mathbf{a}$ on the line. The distance is therefore $\|\mathbf{q} - \mathbf{w}\| = \|\frac{2}{13}\mathbf{a}\| = \frac{2}{\sqrt{13}}$.

1.3.10 **a.** $ax_1 + bx_2 = 0$ (a and b arbitrary real numbers, not both 0)

1.4.1 **b.** $\mathbf{x} = (-2, 0, 1) + x_2(-2, 1, 0)$

1.4.2 **b.**, **c.**, **d.**, **f.**, **g.** are in echelon form; **c.** and **g.** are in reduced echelon form.

1.4.3 **a.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; **e.** $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$,
$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$; **g.** $\begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,
$\mathbf{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

1.4.4 **a.** $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; **c.** inconsistent

1.4.5 **b.** $\mathbf{x} = x_2(0, 1, 0) + x_3(1, 0, 1)$

1.4.8 $\mathbf{x} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$

1.4.10 **a.** $(1, -1, -1, 1)$

1.4.11 **a.** $\mathbf{x} = s(-1, 1, 1, 0) + t(-1, -2, 0, 1)$

1.4.13 Use Proposition 2.1. For each $i = 1, \ldots, m$, we have $\mathbf{A}_i \cdot (c\mathbf{x}) = c(\mathbf{A}_i \cdot \mathbf{x})$ and $\mathbf{A}_i \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A}_i \cdot \mathbf{x} + \mathbf{A}_i \cdot \mathbf{y}$.

1.5.1 $\mathbf{b} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$

1.5.2 **b.** yes; **a.**, **c.** no

1.5.3 **b.** $2b_1 + b_2 - b_3 = 0$; **d.** None.

1.5.7 **b.** $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$; **d.** $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

1.5.8 **a.** 0, 3; **b.** for $\alpha = 0$, $\mathbf{b}$ must satisfy $b_2 = 0$; for $\alpha = 3$, $\mathbf{b}$ must satisfy $b_2 = 3b_1$.

1.5.12 **a.** none, as $A\mathbf{x} = \mathbf{0}$ is always consistent; **b.** take $r = m = n$; **e.** take $r < m$ and $r < n$

1.5.14 **a.** A further hint: By repeatedly doing row operation (iii), we can obtain a row of 0's in the matrix, so the echelon form must have a row of 0's. An alternative approach is this: Because the sum of the rows is $\mathbf{0}$, any vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is consistent must satisfy $b_1 + \cdots + b_m = 0$. (To see this, note that if $A\mathbf{x} = \mathbf{b}$, then this means that $\mathbf{A}_i\mathbf{x} = b_i$, so $\mathbf{0} = (\mathbf{A}_1 + \cdots + \mathbf{A}_m)\mathbf{x} = \mathbf{A}_1\mathbf{x} + \cdots + \mathbf{A}_m\mathbf{x} = b_1 + \cdots + b_m$.) Thus, by Proposition 5.1, we must have $\text{rank}(A) < m$.

1.6.1 64%, 32

1.6.5 $y = 3x^2 - 5x + 1$

1.6.9 center $(-1, 2)$, radius 5

1.6.12 **a.** $(a, b, c, d, e) = (3, 6, 5, 1, 3)$

1.6.13 **a.** $(A, B, C, D) = \frac{1}{2}(1, 1, 3, 3)$

1.6.14 **a.** $I_1 = 3$ amps; **b.** $I_3 = 3$ amps; **c.** $I_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

1.6.16 **b.** No matter what the original cat and mouse populations are, the cats proliferate and the mice die out.

2.1.1 **b.** $\begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$; **f.** $\begin{bmatrix} 5 & 8 \\ 13 & 20 \end{bmatrix}$; **g.** $\begin{bmatrix} 1 & 4 & 5 \\ 3 & 10 & 11 \end{bmatrix}$; **h.** not defined; **k.** $\begin{bmatrix} 4 & 4 \\ 5 & 6 \end{bmatrix}$; **l.** $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$

2.1.5 **a.** As a hint, see part *a* of Exercise 1.2.16.

2.1.7 **b.** Either $A = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$ for some real number β or $A = \alpha \begin{bmatrix} 1 & \beta \\ -1/\beta & -1 \end{bmatrix}$, α any real number, $\beta \neq 0$.

2.2.3 **a.** Since $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, that is the first column of A. For the second column, we need $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - T\left(2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and this is the second column of A. Thus, $A = \begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix}$.

2.2.4 **b.**, **c.**, **e.** yes; **a.**, **d.**, **f.** no

2.2.5 **b.** $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; **e.** $A = \begin{bmatrix} -\frac{2}{5} & -\frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

2.2.6 **a.** S: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, T: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; **b.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; **c.** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

2.3.1 **a.** $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$; **e.** $A^{-1} = \begin{bmatrix} -1 & 3 & -2 \\ -1 & 2 & -1 \\ 2 & -3 & 2 \end{bmatrix}$; **g.** $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$

2.3.2 **b.** $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & -1 & -3 \\ -6 & 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$;

d. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

2.3.3 **a.** BA^2B^{-1}; **b.** BA^nB^{-1}; **c.** assuming A invertible, $BA^{-1}B^{-1}$

2.3.4 Since $A\mathbf{x} = 7\mathbf{x}$, we have $\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}(7\mathbf{x}) = 7(A^{-1}\mathbf{x})$, and so $A^{-1}\mathbf{x} = \frac{1}{7}\mathbf{x}$.

2.3.8 Note that $A(A^2 - 3I) = -I$, so $A(3I - A^2) = I$. Of course, $(3I - A^2)A = I$ as well, so $A^{-1} = 3I - A^2$.

2.4.1 **a.** $\begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$,

$-b_1 + b_2 + b_3 = 0$; **e.** $\begin{bmatrix} 1 & \\ -2 & 1 \end{bmatrix}$, none;

g. $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{bmatrix}$, $b_2 - \frac{1}{2}b_3 + b_4 = 0$

2.4.2 **a.** $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}$; **e.** $\begin{bmatrix} 1 & \\ -2 & 1 \end{bmatrix}$; **g.** $\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 2 & -2 & 1 & \\ 0 & 1 & -\frac{1}{2} & 1 \end{bmatrix}$

2.4.3 **a.** $\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$; **e.** $\begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$;

g. $\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & 2 & 1 & \\ -1 & & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2.4.4 **a.** $L = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 2 & 1 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

b. $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2.4.6 **a.** Solving $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$, we find $\mathbf{y} = (2, 3, -6)$ and $\mathbf{x} = (-8, -1, 6)$; **c.** Solving $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$, we find $\mathbf{y} = (5, 4, -10)$ and $\mathbf{x} = (-11, -2, 10)$.

2.4.10 **a.** Fix i. Let E_1 be the elementary matrix corresponding to adding c_{ji} times row i to row j ($j \neq i$), and let E_2 be the elementary matrix corresponding to adding c_{ki} times row i to row k ($k \neq i, j$). Then multiplying by E_1E_2 first adds c_{ki} times row i to row k and then adds c_{ji} times row i to row j. Since row i stays the same throughout, this is the same as first adding c_{ji} times row i to row j and then adding c_{ki} times row i to row k, i.e., multiplying by E_2E_1. Thus, $E_1E_2 = E_2E_1$. This matrix has 1's on the diagonal, c_{ji} as the ji-entry, c_{ki} as the ki-entry, and 0's elsewhere. In reducing a matrix to echelon form, we can therefore multiply by a *single* matrix to clear out the entries below a pivot.

Similarly, E_1^{-1} has 1's on the diagonal, $-c_{ji}$ as the ji-entry, and 0's elsewhere; then $(E_1E_2)^{-1}$ will have 1's on the diagonal, $-c_{ji}$ as the ji-entry, $-c_{ki}$ as the ki-entry, and 0's elsewhere. We can therefore obtain L by multiplying r matrices, one for each pivot.

2.5.1 **b.** $\begin{bmatrix} 0 & 0 \\ 5 & 5 \end{bmatrix}$; **e.** $\begin{bmatrix} 1 & 5 & 7 \\ 2 & 8 & 10 \end{bmatrix}$; **g.** $\begin{bmatrix} 1 & 3 \\ 4 & 10 \\ 5 & 11 \end{bmatrix}$; **j.** $\begin{bmatrix} 6 & 4 \\ 4 & 5 \end{bmatrix}$; **k.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & 5 \end{bmatrix}$

2.5.2 **a.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$; **b.** $\begin{bmatrix} 6 \end{bmatrix}$; **f.** $\begin{bmatrix} 5 \end{bmatrix}$

2.5.3 **b.** $\dfrac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}$

2.5.5 $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}} = BA$; thus $(AB)^{\mathsf{T}} = AB$ if and only if $BA = AB$.

2.5.15 Since $(A^{\mathsf{T}}A)\mathbf{x} = \mathbf{0}$, we have $(A^{\mathsf{T}}A)\mathbf{x} \cdot \mathbf{x} = 0$. By Proposition 5.2, $0 = A^{\mathsf{T}}(A\mathbf{x}) \cdot \mathbf{x} = A\mathbf{x} \cdot A\mathbf{x} = \|A\mathbf{x}\|^2$, and so $\|A\mathbf{x}\| = 0$. This means that $A\mathbf{x} = \mathbf{0}$.

2.5.17 A hint: It suffices to see why $A_\theta\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A_\theta^{-1}\mathbf{y}$. Since rotation doesn't change the length of vectors, we only need to see that the angle between $A_\theta\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $\mathbf{x}$ and $A_\theta^{-1}\mathbf{y}$.

2.5.19 **d.** By c., every orthogonal matrix can be written either as A_θ or as $A_\theta\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.5.20 **b.** If A is orthogonal, then $A^{-1} = A^{\mathsf{T}}$, and so $(A^{-1})^{\mathsf{T}}(A^{-1}) = (A^{\mathsf{T}})^{\mathsf{T}}A^{\mathsf{T}} = AA^{\mathsf{T}} = I$ by part *e* of Exercise 19.

3.1.1 **b.**, **e.**, **g.**, **h.** yes; **a.**, **c.**, **d.**, **f.**, **i.** no

3.1.2 **d.** yes; **a.**, **b.**, **c.** no

3.1.3 The argument is valid only if there is *some* vector in the subspace. The first criterion is equivalent to the subspace's being nonempty.

3.1.9 **a.** $\text{Span}\big((1,1,1)\big)$; **d.** $\{\mathbf{0}\}$

3.1.10 If $\mathbf{v} \in V \cap V^\perp$, then $\mathbf{v}\cdot\mathbf{v} = 0$, so $\mathbf{v} = \mathbf{0}$. Moreover, $\mathbf{0} \in V \cap V^\perp$, so $V \cap V^\perp = \{\mathbf{0}\}$.

3.1.12 Suppose $\mathbf{v} \in V$. Then $\mathbf{v}\cdot\mathbf{w} = 0$ for every $\mathbf{w} \in W$, so, by definition, $\mathbf{v} \in W^\perp$.

3.2.1 Let's show that $\mathbf{R}(B) \subset \mathbf{R}(A)$ if B is obtained by performing any row operation on A. Obviously, a row interchange doesn't affect the span. If $\mathbf{B}_i = c\mathbf{A}_i$ and all the other rows are the same, $c_1\mathbf{B}_1 + \cdots + c_i\mathbf{B}_i + \cdots + c_m\mathbf{B}_m = c_1\mathbf{A}_1 + \cdots + (c_ic)\mathbf{A}_i + \cdots + c_m\mathbf{A}_m$, so any vector in $\mathbf{R}(B)$ is in $\mathbf{R}(A)$. If $\mathbf{B}_i = \mathbf{A}_i + c\mathbf{A}_j$ and all the other rows are the same, then $c_1\mathbf{B}_1 + \cdots + c_i\mathbf{B}_i + \cdots + c_m\mathbf{B}_m = c_1\mathbf{A}_1 + \cdots + c_i(\mathbf{A}_i + c\mathbf{A}_j) + \cdots + c_m\mathbf{A}_m = c_1\mathbf{A}_1 + \cdots + c_i\mathbf{A}_i + \cdots + (c_j + cc_i)\mathbf{A}_j + \cdots + c_m\mathbf{A}_m$, so once again any vector in $\mathbf{R}(B)$ is in $\mathbf{R}(A)$.

To see that $\mathbf{R}(A) \subset \mathbf{R}(B)$, we observe that the matrix A is obtained from B by performing the (inverse) row operation (this is why we need $c \neq 0$ for the second type of row operation). Since $\mathbf{R}(B) \subset \mathbf{R}(A)$ and $\mathbf{R}(A) \subset \mathbf{R}(B)$, we have $\mathbf{R}(A) = \mathbf{R}(B)$.

3.2.2 **a.** $\mathbf{C}(A) = \{\mathbf{b} : 2b_1 - b_2 = 3b_1 + b_3 = 0\}$; **b.** $\mathbf{C}(A) = \mathbb{R}^3$

3.2.3 **a.** $X = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, Y any $2 \times n$ matrix whose columns are scalar multiples of $(1, 3)$

3.2.4 **a.** $b_1 - b_2 - b_3 = 0$; **b.** Since $\mathbf{N}(A^\mathsf{T}) = \mathbf{C}(A)^\perp$, we need to find $\mathbf{C}(A)^\perp$. From **a.** we know that $(1, -1, -1) \in \mathbf{C}(A)^\perp$, but does it span? If we had $\mathbf{c} \in \mathbf{C}(A)^\perp$ that were not a multiple of $(1, -1, -1)$, then we would have a new constraint $\mathbf{c}\cdot\mathbf{b} = 0$ for a vector $\mathbf{b}$ to lie in $\mathbf{C}(A)$. Then $\mathbf{C}(A)$ would be at most a line, rather than the plane we obtained in **a.**

3.2.5 $\mathbf{R}(A)$: $(1, 2, 1, 1), (0, 0, 2, -2)$; $\mathbf{C}(A)$: $(1, 2, 1), (0, 1, -1)$;
$\mathbf{N}(A)$: $(-2, 1, 0, 0), (-2, 0, 1, 1)$

3.2.6 **b.** $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

3.2.8 **a.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3.2.9 $\Longrightarrow$: Recall that μ_A is one-to-one if $\mu_A(\mathbf{x}) = \mu_A(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$. Assume μ_A is one-to-one and that $\mathbf{x} \in \mathbf{N}(A)$. Then $\mu_A(\mathbf{x}) = \mathbf{0} = \mu_A(\mathbf{0})$, so $\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{N}(A) = \{\mathbf{0}\}$.
$\Longleftarrow$: Suppose $\mathbf{N}(A) = \{\mathbf{0}\}$ and $\mu_A(\mathbf{x}) = \mu_A(\mathbf{y})$. Then $\mu_A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, so $\mathbf{x} - \mathbf{y} \in \mathbf{N}(A)$, from which we conclude that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, i.e., $\mathbf{x} = \mathbf{y}$. This shows that μ_A is one-to-one.

3.3.2 **a.**, **b.**, **d.**, **e.** yes; **c.**, **f.** no

3.3.3 **d.** yes; **a.**, **b.**, **c.** no

3.3.4 **b.** $(0, 2, -1)$; **d.** $(2, -1, 1, 0)$

3.3.5 **a.** $\mathbf{R}(A)$: $(3, -1)$, $\mathbf{C}(A)$: $(1, 2, -3)$, $\mathbf{N}(A)$: $(1, 3)$, $\mathbf{N}(A^{\mathsf{T}})$: $(2, -1, 0)$, $(3, 0, 1)$; **c.** $\mathbf{R}(A)$: $(1, 0, 2)$, $(0, 1, -1)$, $\mathbf{C}(A)$: $(1, 1, 1, 1)$, $(1, 2, 1, 0)$, $\mathbf{N}(A)$: $(-2, 1, 1)$, $\mathbf{N}(A^{\mathsf{T}})$: $(1, 0, -1, 0)$, $(-2, 1, 0, 1)$

3.3.6 $\{(-3, 2, 1, 0), (-2, 0, 0, 1)\}$

3.3.13 Suppose there is a nontrivial linear combination

$$(*) \qquad \mathbf{0} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k,$$

where *some* $d_i \neq 0$. Given any vector $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$, add the equation for $\mathbf{0}$ to obtain $\mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_k + d_k)\mathbf{v}_k$. These are distinct expressions since some $d_i \neq 0$. We can obtain infinitely many distinct expressions by multiplying the equation $(*)$ by arbitrary scalars $s \in \mathbb{R}$.

3.3.14 Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_\ell\mathbf{v}_\ell = \mathbf{0}$. Then we have $c_1\mathbf{v}_1 + \cdots + c_\ell\mathbf{v}_\ell + 0\mathbf{v}_{\ell+1} + \cdots + 0\mathbf{v}_k = \mathbf{0}$, so linear independence of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ tells us that all the coefficients must be 0. That is, $c_1 = \cdots = c_\ell = 0$, as required.

3.3.23 A hint: Use the definition of $U + V$ to show that the vectors span. To establish linear independence, suppose $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + d_1\mathbf{v}_1 + \cdots + d_\ell\mathbf{v}_\ell = \mathbf{0}$. Then what can you say about the vector $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = -(d_1\mathbf{v}_1 + \cdots + d_\ell\mathbf{v}_\ell)$?

3.4.1 **a.** $\{(1, 2, 3), (3, 4, 7)\}$, dim 2

3.4.3 **f.** $\mathbf{R}(A)$: $\{(1, 0, -1, 2, 0, 1), (0, 1, 1, 3, 0, -2), (0, 0, 0, 0, 1, -1)\}$,
$\mathbf{C}(A)$: $\{(1, 0, -1, 0), (1, 1, 2, 4), (0, -2, 1, -1)\}$,
$\mathbf{N}(A)$: $\{(1, -1, 1, 0, 0, 0), (-2, -3, 0, 1, 0, 0), (-1, 2, 0, 0, 1, 1)\}$,
$\mathbf{N}(A^{\mathsf{T}})$: $\{(1, 1, 1, -1)\}$

3.4.5 **a.** $\{(-3, -2, 1, 0), (-4, 5, 0, 1)\}$; **b.** $\{(1, 0, 3, 4), (0, 1, 2, -5)\}$

3.4.8 **a.** $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}$; **c.** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

3.4.9 **b.** $\mathbf{R}(A)$: $\{(1, 1, 2, 1), (0, 0, 2, -1)\}$, $\mathbf{N}(A)$: $\{(-1, 1, 0, 0), (-4, 0, 1, 2)\}$,
$\mathbf{C}(A)$: $\{(1, 1, -2), (0, 1, 0)\}$, $\mathbf{N}(A^{\mathsf{T}})$: $\{(2, 0, 1)\}$

3.4.10 **b.** $\begin{bmatrix} \frac{1}{3}b_1 \\ \frac{1}{3}b_1 + \frac{1}{2}b_2 \\ \frac{1}{3}b_1 - \frac{1}{2}b_2 \end{bmatrix}$

3.4.13 Since U is a matrix in echelon form, its *last* $m - r$ rows are $\mathbf{0}$. When we consider the matrix product $A = BU$, we see that every column of A is a linear combination of the *first* r columns of B; hence, these r column vectors span $\mathbf{C}(A)$. Since $\dim \mathbf{C}(A) = r$, these column vectors must give a basis (see Proposition 4.4).

3.4.14 **a.** Calculate dimensions: If $\dim V = k$, then $\dim V^\perp = n - k$, and $\dim(V^\perp)^\perp = n - (n - k) = k = \dim V$. Since we know that $V \subset (V^\perp)^\perp$, applying Proposition 4.3, we deduce that $V = (V^\perp)^\perp$.

3.5.1 $\dim \mathbf{N}(A) = 1$ because the graph is connected; $\dim \mathbf{N}(A^{\mathsf{T}}) = 2$ because there are two independent loops; $\dim \mathbf{R}(A) = 5 - \dim \mathbf{N}(A) = 4$; $\dim \mathbf{C}(A) = 6 - \dim \mathbf{N}(A^{\mathsf{T}}) = 4$.

3.5.3 $\dim \mathbf{N}(A) = 1$ because the graph is connected; $\dim \mathbf{N}(A^{\mathsf{T}}) = 3$ because there are three independent loops; $\dim \mathbf{R}(A) = 4 - \dim \mathbf{N}(A) = 3$; $\dim \mathbf{C}(A) = 6 - \dim \mathbf{N}(A^{\mathsf{T}}) = 3$.

3.6.2 **a.**, **c.**, **e.** yes; **b.**, **d.**, **f.** no

3.6.3 **a.**, **f.**, **g.** yes; **b.**, **c.**, **d.**, **e.** no

3.6.4 mn

3.6.5 $\dim \mathcal{U} = \dim \mathcal{L} = \frac{1}{2}n(n+1)$, $\dim \mathcal{D} = n$

3.6.6 **a.** not a subspace since $\mathbf{0}$ does not have this property; **c.** a one-dimensional subspace with basis $\{e^{-2t}\}$; **f.** a two-dimensional subspace with basis $\{\cos t, \sin t\}$: show that if f lies in this subspace and has the properties $f(0) = a$ and $f'(0) = b$, then $f(t) = a\cos t + b\sin t$ (Hint: Consider $g(t) = f(t) - a\cos t - b\sin t$ and show that $h(t) = (g(t))^2 + (g'(t))^2$ is a constant); **g.** Guess two linearly independent exponential solutions, and it will follow from Theorem 3.4 of Chapter 7 that these form a basis.

3.6.10 **b.** $\{I_n\}$

3.6.14 **a.** $f(t) = t$ gives a basis; **b.** $f(t) = t - \frac{1}{2}$ gives a basis.

3.6.15 **b.** $f(t) = t^2 - t + \frac{1}{6}$ gives a basis.

3.6.16 $f(t) = 19 - 112t + 110t^2$ gives a basis.

3.6.18 Hint: Use the addition formulas for sin and cos to derive the formulas

$$\sin kt \sin \ell t = \tfrac{1}{2}\big(\cos(k-\ell)t - \cos(k+\ell)t\big),$$
$$\sin kt \cos \ell t = \tfrac{1}{2}\big(\sin(k+\ell)t - \sin(k-\ell)t\big).$$

3.6.20 **a.** Suppose $\mathbf{x}$ and $\mathbf{y}$ are in the given subset. Then there are constants C and D so that $|x_k| \le C$ and $|y_k| \le D$ for all k; thus $|x_k + y_k| \le |x_k| + |y_k| \le C + D$ for all k, so $\mathbf{x}+\mathbf{y}$ is in the subset. (Note we've used the triangle inequality, Exercise 1.2.18.) And $|cx_k| = |c||x_k| \le |c|C$ for all k, so $c\mathbf{x}$ is in the subset. Since $\mathbf{0}$ is obviously in the subset, it must be a subspace.

4.1.1 **b.** $(-1, 0, 1, 3)$

4.1.3 **a.** $V^\perp$ is spanned by $\mathbf{a} = (1,1,2)$, so $P_{V^\perp} = \dfrac{1}{\|\mathbf{a}\|^2}\mathbf{a}\mathbf{a}^\top = \dfrac{1}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}1 & 1 & 2\end{bmatrix} =$

$$\frac{1}{6}\begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{bmatrix}; \text{ so } P_V = I - P_{V^\perp} = \frac{1}{6}\begin{bmatrix}5 & -1 & -2\\-1 & 5 & -2\\-2 & -2 & 2\end{bmatrix}.$$

b. Let $A = \begin{bmatrix}1 & -2\\1 & 0\\-1 & 1\end{bmatrix}$. Then $A^\top A = \begin{bmatrix}3 & -3\\-3 & 5\end{bmatrix}$, and so $P_V =$

$$A(A^\top A)^{-1}A^\top = \frac{1}{6}\begin{bmatrix}1 & -2\\1 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}5 & 3\\3 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\-2 & 0 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5 & -1 & -2\\-1 & 5 & -2\\-2 & -2 & 2\end{bmatrix}.$$

4.1.6 $\overline{\mathbf{x}} = \dfrac{1}{14}\begin{bmatrix}1\\4\end{bmatrix}$; $\dfrac{1}{14}\begin{bmatrix}5\\6\\-3\end{bmatrix}$.

4.1.9 **a.** Fitting $y = a$ to the data yields the inconsistent system $Aa = \mathbf{b}$, with $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}$. Then $A^{\mathsf{T}}A = [4]$ and $A^{\mathsf{T}}\mathbf{b} = [9]$, so $\overline{a} = 9/4$. (Notice this is just the average of the given y-values.) As the theory predicts, the sum of the errors is $(0 - \frac{9}{4}) + (1 - \frac{9}{4}) + (3 - \frac{9}{4}) + (5 - \frac{9}{4}) = 0$; **c.** $a = \frac{1}{4}$, $b = \frac{29}{20}$, $c = \frac{23}{20}$.

4.1.11 $a \approx 1.866$, $k \approx 0.878$.

4.1.13 Suppose $\text{proj}_V \mathbf{x} = \mathbf{p}$ and $\text{proj}_V \mathbf{y} = \mathbf{q}$. Then $\mathbf{x} - \mathbf{p}$ and $\mathbf{y} - \mathbf{q}$ are vectors in $V^\perp$. Then

$$\mathbf{x} + \mathbf{y} = (\mathbf{p} + \mathbf{q}) + \big((\mathbf{x} + \mathbf{y}) - (\mathbf{p} + \mathbf{q})\big) = \underbrace{(\mathbf{p} + \mathbf{q})}_{\in V} + \underbrace{\big((\mathbf{x} - \mathbf{p}) + (\mathbf{y} - \mathbf{q})\big)}_{\in V^\perp},$$

so $\text{proj}_V(\mathbf{x} + \mathbf{y}) = \mathbf{p} + \mathbf{q}$, as required. Similarly, since

$$c\mathbf{x} = c\big(\mathbf{p} + (\mathbf{x} - \mathbf{p})\big) = (c\mathbf{p}) + (c\mathbf{x} - c\mathbf{p}) = \underbrace{(c\mathbf{p})}_{\in V} + \underbrace{c(\mathbf{x} - \mathbf{p})}_{\in V^\perp},$$

we infer that $\text{proj}_V(c\mathbf{x}) = c\mathbf{p}$, as well.

4.1.17 **a.** $1/\sqrt{6}$

4.2.2 **c.** $\mathbf{q}_1 = \frac{1}{\sqrt{2}}(1, 0, 1, 0)$, $\mathbf{q}_2 = \frac{1}{2}(1, 1, -1, 1)$, $\mathbf{q}_3 = \frac{1}{\sqrt{2}}(0, 1, 0, -1)$

4.2.4 **a.** $\mathbf{w}_1 = (1, 3, 1, 1)$, $\mathbf{w}_2 = \frac{1}{2}(1, -1, 1, 1)$, $\mathbf{w}_3 = (-2, 0, 1, 1)$; **b.** $\text{proj}_V(4, -1, 5, 1) = (4, -1, 3, 3)$; **c.** $P_V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

4.2.5 **a.** $\mathbf{w}_1 = (1, -1, 0, 2)$, $\mathbf{w}_2 = \frac{1}{2}(1, 1, 2, 0)$; **b.** $\mathbf{p} = (1, -1, 0, 2)$; **c.** $\overline{\mathbf{x}} = (1, 0)$

4.2.7 **b.** Since $\text{rank}(A) = 2$, we know that $\mathbf{C}(A) = \mathbb{R}^2$. Row reducing the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & -1 & b_2 \end{array}\right]$ yields one solution (of many possible) $\mathbf{v} = (b_1 - b_2, b_2, 0)$. The key point is that the unique solution lying in the row space is obtained by projecting an *arbitrary* solution onto $\mathbf{R}(A)$. The rows of A are orthogonal, so to find the solution in the row space, we take $\mathbf{x} = \text{proj}_{\mathbf{R}(A)}\mathbf{v} = \frac{\mathbf{v}\cdot(1,1,1)}{\|(1,1,1)\|^2}(1, 1, 1) + \frac{\mathbf{v}\cdot(0,1,-1)}{\|(0,1,-1)\|^2}(0, 1, -1) = \frac{b_1}{3}(1, 1, 1) + \frac{b_2}{2}(0, 1, -1) = (\frac{1}{3}b_1, \frac{1}{3}b_1 + \frac{1}{2}b_2, \frac{1}{3}b_1 - \frac{1}{2}b_2)$.

4.2.8 **a.** $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$, $R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$; **b.** $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$,

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}$$

4.2.11 **b.** The i^{th} row of A^{-1} is $\mathbf{a}_i^{\mathsf{T}}/\|\mathbf{a}_i\|^2$.

4.2.12 **a.** $\{1, t\}$, $(\text{proj}_V f)(t) = \frac{4}{3} - t$; **d.** $\{1, \cos t, \sin t\}$, $(\text{proj}_V f)(t) = 2\sin t$

4.3.1 **a.** Rotating the vector $\mathbf{e}_1$ by $-\pi/4$ gives the vector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; reflecting that vector across the line $x_1 = x_2$ gives $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Similarly, rotating $\mathbf{e}_2$ by $-\pi/4$ gives the vector $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is left unchanged by the reflection. Thus, the standard matrix for T is $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

4.3.3 **a.** This symmetry carries $\mathbf{e}_1$ to $\mathbf{e}_2$, carries $\mathbf{e}_2$ to $-\mathbf{e}_1$, and leaves $\mathbf{e}_3$ fixed. Thus, the standard matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since the columns of this matrix form an orthonormal set, the matrix is orthogonal.

4.3.5 **a.** The change-of-basis matrix is $P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, whose inverse is $P^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. Thus, $[T]_{\mathcal{B}} = P^{-1}[T]_{\text{stand}}P = \begin{bmatrix} 36 & 24 \\ -55 & -37 \end{bmatrix}$.

4.3.7 $\begin{bmatrix} -7 & 24 & 8 \\ -4 & 14 & 5 \\ 5 & -17 & -6 \end{bmatrix}$

4.3.11 $\frac{1}{9}\begin{bmatrix} 7 & 4 & -4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix}$

4.3.14 $\begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4.3.16 $\frac{1}{17}\begin{bmatrix} 3 & 1 & 5 & 4 \\ 1 & 6 & -4 & 7 \\ 5 & -4 & 14 & 1 \\ 4 & 7 & 1 & 11 \end{bmatrix}$

4.3.24 **a.** $\mathbf{v}_1 = (1, 2, 1)$; **b.** $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1, 0, 1)$, $\mathbf{v}_3 = \frac{1}{\sqrt{3}}(1, -1, 1)$; **c.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$; **d.** T is a rotation of $-\pi/2$ around the line spanned by $(1, 2, 1)$ (as viewed from high above that vector).

4.3.26 **a.** If we write $\mathbf{x} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3$, the equation of the curve of intersection becomes $y_1^2 + \sin^2\phi\, y_2^2 = 1$, $y_3 = 0$.

4.4.1 **a.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\ker(T) = \{O\}$, image $(T) = \mathcal{M}_{2\times 2}$;

b. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$, $\ker(T) = \{O\}$, image $(T) = \mathcal{M}_{2\times 2}$

4.4.2 **a.** $\begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n \end{bmatrix}$

4.4.3 Let $f, g \in \mathcal{P}_3$. Then, using the usual differentiation rules, $T(f+g)(t) = (f+g)''(t) + 4(f+g)'(t) - 5(f+g)(t) = (f''(t) + g''(t)) + 4(f'(t) + g'(t)) - 5(f(t) + g(t)) = \big(f''(t) + 4f'(t) - 5f(t)\big) + \big(g''(t) + 4g'(t) - 5g(t)\big) = T(f)(t) + T(g)(t)$, so $T(f+g) = T(f) + T(g)$. For any scalar c, we have $T(cf)(t) = (cf)''(t) + 4(cf)'(t) - 5(cf)(t) = cf''(t) + 4cf'(t) - 5cf(t) = c\big(f''(t) + 4f'(t) - 5f(t)\big) = cT(f)(t)$, so $T(cf) = cT(f)$. Thus, T is a linear transformation.

To compute the matrix A, we need to apply T to each of the basis vectors $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = t$, $\mathbf{v}_3 = t^2$, $\mathbf{v}_4 = t^3$: $T(\mathbf{v}_1) = -5 = -5\mathbf{v}_1$, $T(\mathbf{v}_2) = 4 - 5t = 4\mathbf{v}_1 - 5\mathbf{v}_2$, $T(\mathbf{v}_3) = 2 + 4(2t) - 5t^2 = 2\mathbf{v}_1 + 8\mathbf{v}_2 - 5\mathbf{v}_3$, and $T(\mathbf{v}_4) = 6t + 4(3t^2) - 5t^3 = 6\mathbf{v}_2 + 12\mathbf{v}_3 - 5\mathbf{v}_4$. Thus, the matrix is as given in the text.

4.4.4 **a.** We use the matrix A from Example 5 and apply Theorem 4.2. Since $\mathcal{W} = \mathcal{W}'$, we have $Q = I$. The change-of-basis matrix from $\mathcal{V}$ to $\mathcal{V}'$ is $P = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. So $[T]_{\mathcal{V},\mathcal{W}} = Q^{-1}AP = AP = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

Note that this checks with $T(1) = 0$, $T(t-1) = 1$, $T((t-1)^2) = 2(t-1)$, and $T((t-1)^3) = 3(t-1)^2$.

4.4.8 **a.** $T\big(\mathbf{u} + t(\mathbf{v} - \mathbf{u})\big) = T(\mathbf{u}) + tT(\mathbf{v} - \mathbf{u}) = T(\mathbf{u}) + t\big(T(\mathbf{v}) - T(\mathbf{u})\big)$

4.4.14 **a.** no; **b.** $\ker(T) = \text{Span}\,(1 - 2t, 1 - 3t^2, 1 - 4t^3)$, image $(T) = \mathbb{R}$; **d.** $\ker(T) = \{\mathbf{0}\}$, image $(T) = \{g \in \mathcal{P} : g(0) = 0\}$

5.1.1 **b.** -4; **d.** 6

5.1.8 **c.** $\prod\limits_{i<j}(t_i - t_j)$, where $\prod$ denotes product.

5.1.10 $\det A = \pm 1$

5.2.1 **b.** -4; **d.** 6

5.2.2 **a.** $-4/3$; **b.** $\begin{bmatrix} 2 & 0 & -1 \\ -\frac{4}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

5.2.3 $-3; \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ -\frac{4}{3} & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$

5.2.5 **a.** $t^2 - 5t - 6$; **e.** $t^2 - 4t + 4$; **f.** $-t^3 + 9t$; **j.** $-t^3 + 4t^2 - t - 6$

5.3.1 **a.** -7

5.3.2 **a.** -2

6.1.1 **a.** eigenvalues $-1, 6$; eigenvectors $(-5, 2), (1, 1)$;
f. eigenvalues 2, 2; only eigenvector $(1, 1)$;
g. eigenvalues $-3, 0, 3$; eigenvectors $(-1, 0, 1), (1, -1, 1), (1, 2, 1)$;
j. eigenvalues 1, 1, 2; eigenvectors $(0, 1, 0), (1, 0, 0)$;
n. eigenvalues $-1, 2, 3$; eigenvectors $(1, 1, 1), (2, 1, 2), (-1, 1, 1)$

6.1.13 **b.** eigenvectors t^k, $k = 0, 1, 2, 3$, with corresponding eigenvalue k

6.1.15 **a.** For any real number λ, $f(t) = t^\lambda$ spans $\mathbf{E}(\lambda)$.

6.1.16 Hint for **b.**: Prove $p_{AB}(t^2) = p_{BA}(t^2)$ by considering the matrix $\left[\begin{array}{c|c} tI & A \\ \hline B & tI \end{array}\right]$ and multiplying on the left by either $\left[\begin{array}{c|c} tI & -A \\ \hline O & I \end{array}\right]$ or $\left[\begin{array}{c|c} I & O \\ \hline -B & tI \end{array}\right]$.

6.2.1 **a.**, **g.**, **n.** diagonalizable; **f.**, **j.** not diagonalizable

6.2.4 A hint: See Exercise 17.

6.2.7 **a.** O; **b.** $(2, 3)$; $\mathbf{C}(A - 2I) = \mathbf{N}(A - 2I)$ because of part **a.**; **c.** $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

6.2.8 $\cos\theta \pm i \sin\theta$, eigenvectors $(1, \mp i)$

6.3.2 4 : 3

6.3.5 2/3; 5/6

6.3.6 9/13

6.3.7 $a_k = 2^k + 1$

6.3.11 $a_k = \frac{1}{3}\left(2^k + (-1)^{k+1}\right)$

6.3.12 **b.** $\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix} = (c_0 + 2m_0)(1.1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (c_0 + m_0)(1.2)^k \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, so—no matter what the original cat and mouse populations—the cats proliferate and the mice die out.

6.4.1 **a.** $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$; **d.** $\begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$; **e.** $\frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

6.4.2 $(2, 2, 4)$

6.4.7 There is an orthogonal matrix Q so that $Q^{-1}AQ = \Lambda = \lambda I$. But then $A = Q(\lambda I)Q^{-1} = \lambda I$.

6.4.16 **b.** ellipse $y_1^2 + 2y_2^2 = 2$, where $\mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x}$;

c. parabola $y_1 = 5y_2^2 - 1$, where $\mathbf{y} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\mathbf{x}$.

6.4.17 **a.** hyperboloid of 1 sheet $-2y_1^2 + y_2^2 + 4y_3^2 = 4$,

where $\mathbf{y} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}^{\mathsf{T}} \mathbf{x}$; **d.** hyperbolic paraboloid (saddle surface)

$-y_1^2 + 3y_3^2 + \sqrt{3}y_2 = 1$, where $\mathbf{y} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}^{\mathsf{T}} \mathbf{x}$.

7.1.2 **b.** $(1, 1+i)$, $(1, 1-i)$, $\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$; $(1, 1)$, $(0, -1)$, $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$;

c. $(1, -1, 1)$, $(1, i, 0)$, $(1, -i, 0)$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$,

$(1, -1, 1)$, $(1, 0, 0)$, $(0, -1, 0)$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

7.1.5 Suppose, as in the proof, that $(A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ gives a basis for $\mathbf{N}(A - \lambda I)$. Suppose now that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Multiplying by $A - \lambda I$ gives $c_3\mathbf{v}_2 = \mathbf{0}$, and so we must have $c_3 = 0$. But since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be a linearly independent set, we now infer that $c_1 = c_2 = 0$ as well.

7.1.7 **a.** Possible Jordan canonical forms of A are

when $\dim \mathbf{E}(\lambda) = 2$, $\dim \mathbf{E}(\mu) = 2$: $\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{bmatrix}$,

when $\dim \mathbf{E}(\lambda) = 1$, $\dim \mathbf{E}(\mu) = 2$: $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{bmatrix}$,

when $\dim \mathbf{E}(\lambda) = 2$, $\dim \mathbf{E}(\mu) = 1$: $\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & 1 \\ & & & \mu \end{bmatrix}$,

when $\dim \mathbf{E}(\lambda) = 1$, $\dim \mathbf{E}(\mu) = 1$: $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \mu & 1 \\ & & & \mu \end{bmatrix}$.

7.1.8 **a.** $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$;

d. $J = \left[\begin{array}{ccc|c} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ \hline & & & 2 \end{array}\right]$, $P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

7.2.1 **a.** $\Psi = \left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 0 & 2 \\ \hline 0 & 0 & 1 \end{array}\right]$, a rotation through angle $\pi/2$ about the point $(-2, 0)$;

b. $\Psi = \left[\begin{array}{cc|c} 0 & -1 & 2 \\ -1 & 0 & 2 \\ \hline 0 & 0 & 1 \end{array}\right]$, a reflection about the line $x_1 + x_2 = 2$

7.2.2 **b.** glide reflection: reflect about the line $x_2 = \frac{1}{2}$ and translate by $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

7.2.3 **a.** If $\Psi = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{a} \\ \hline 0 & 0 & 1 \end{array}\right]$, then $\Psi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\Psi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}$. Since A must be orthogonal, $\Psi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$. Thus, there are four possible isometries: the identity, reflection across the x_1-axis, reflection across the x_2-axis, and rotation by π.

7.2.10 **a.** It is easiest to show that we can take the points 0, 1, and -1, say, to any three points, a, b, and c. (Then by composition of functions we can take P, Q, R to 0, 1, -1 and then to P', Q', R'.) In terms of homogeneous coordinates, then, we want $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to go to $\begin{bmatrix} a \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to multiple of $\begin{bmatrix} b \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to some multiple of $\begin{bmatrix} c \\ 1 \end{bmatrix}$. Some straightforward algebra leads to $\Psi = \begin{bmatrix} \lambda b - a & a \\ \lambda - 1 & 1 \end{bmatrix}$ with $\lambda = 2\frac{c-a}{c-b}$.

7.2.12 **c.** $f(x_1, x_2, x_3) = \frac{1}{\sqrt{a_1^2 + a_2^2}\left(\mathbf{a} \cdot (\mathbf{x} - \mathbf{a})\right)} \begin{bmatrix} \|\mathbf{a}\|^2 (a_2 x_1 - a_1 x_2) \\ \|\mathbf{a}\| \left(a_1 a_3 x_1 - a_1^2 x_3 + a_2 (a_3 x_2 - a_2 x_3)\right) \end{bmatrix}$

7.3.1 **a.** $-e^{-t} \begin{bmatrix} -5 \\ 2 \end{bmatrix} + e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; **d.** $e^{2t}\left((2 - 3t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$; **e.** $e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

7.3.2 **a.** $(-\cos t + \sin t) \begin{bmatrix} -5 \\ 2 \end{bmatrix} + (e^{\sqrt{6}t} + e^{-\sqrt{6}t}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; **d.** $\begin{bmatrix} \frac{1}{6}t^3 + t^2 + 2t + 1 \\ t + 2 \end{bmatrix}$

7.3.3 **c.** normal modes $e^{\pm it}\begin{bmatrix}1\\1\end{bmatrix}$ and $e^{\pm 2it}\begin{bmatrix}2\\-1\end{bmatrix}$

7.3.4 $\begin{bmatrix}e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t}\\ 0 & e^{2t} & te^{2t}\\ 0 & 0 & e^{2t}\end{bmatrix}$

7.3.5 **a.** $y(t) = e^{2t} - 2e^{-t}$; **b.** $y(t) = e^t + te^t$

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LIST OF BLUE BOXES

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